Chapter 4

Applications of First-order Differential Equations to Real World Systems

In Section 1.4 we have seen that real world problems can be represented by first-order differential equations.

In chapter 2 we have discussed few methods to solve first order differential equations. We solve in this chapter first-order differential equations modeling phenomena of cooling, population growth, radioactive decay, mixture of salt solutions, series circuits, survivability with AIDS, draining a tank, economics and finance, drug distribution, pursuit problem and harvesting of renewable natural resources.

4.1 Cooling/Warming law

We have seen in Section 1.4 that the mathematical formulation of Newton's empirical law of cooling of an object in given by the linear first-order differential equation (1.17)

$$\frac{dT}{dt} = \alpha(T - T_m)$$

This is a separable differential equation. We have

$$\frac{dT}{(T - T_m)} = \alpha dt$$

or $\ln|T-T_m| = \alpha t + c_1$

or $T(t) = T_m + c_2 e^{\alpha t}$ (4.1)

Example 4.1: When a chicken is removed from an oven, its temperature is measured at 300^0F. Three minutes later its temperature is 200^o F. How long will it take for the chicken to cool off to a room temperature of 70^oF.

Solution: In (4.1) we put $T_m = 70$ and T=300 at for t=0.

$T(0)=300=70+c_2 e^{\alpha.0}$

This gives $c_2=230$

For t=3, T(3)=200

Now we put t=3, T(3)=200 and $c_2=230$ in (4.1) then

$200=70 + 230\, e^{\alpha.3}$

or $e^{3\alpha} = \frac{130}{230}$

or $3\alpha = \ln\frac{13}{23}$

or $\alpha = \frac{1}{3}\ln\frac{13}{23} = -0.19018$

Thus $T(t)=70+230\, e^{-0.19018t}$ (4.2)

We observe that (4.2) furnishes no finite solution to T(t)=70 since

$$\lim_{t\to\infty} T(t) = 70.$$

The temperature variation is shown graphically in Figure 4.1. We observe that the limiting temperature is 70^0F.

Figure 4.1

4.2 Population Growth and Decay

We have seen in section 1.4.1 that the differential equation

$$\frac{dN(t)}{dt} = kN(t)$$

where N(t) denotes population at time t and k is a constant of proportionality, serves as a model for population growth and decay of insects, animals and human population at certain places and duration.

Solution of this equation is

$N(t)=Ce^{kt}$, where C is the constant of integration:

$$\frac{dN(t)}{N(t)} = kdt$$

Integrating both sides we get

$\ln N(t) = kt + \ln C$

or $\ln \frac{N(t)}{C} = kt$

or $N(t) = Ce^{kt}$

C can be determined if N(t) is given at certain time.

Example 4.2: The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 years, how long it will take to triple?

Solution : Let N(t) denote the population at time t. Let N(0) denote the initial population (population at t=0).

$$\frac{dN}{dt} = kN(t)$$

Solution is $N(t) = Ae^{kt}$, where A=N(0)

$Ae^{6k} = N(6) = 2N(0) = 2A$

or $e^{6k} = 2$ or $k = \frac{1}{6} \ln 2$

Find t when N(t)=3A=3N(0)

or $N(0)\, e^{kt} = 3N(0)$

or $3 = e^{\frac{1}{6}(\ln 2)t}$

or $\ln 3 = \frac{(\ln 2)t}{6}$

or $t = \frac{6 \ln 3}{\ln 2} \simeq 9.6$ years (approximately 9 years 6 months)

Example 4.3 Let population of country be decreasing at the rate proportional to its population. If the population has decreased to 25% in 10 years, how long will it take to be half?

Solution:_ This phenomenon can be modeled by $\frac{dN}{dt} = kN(t)$

Its solution is

$N(t)=N(0)\, e^{kt}$, where

N(0) in the initial population

For t=10, $N(10)=\frac{1}{4}N(0)$

$\frac{1}{4}N(0) = N(0)\, e^{10k}$

or $e^{10k}=\frac{1}{4}$

or $k=\frac{1}{10}\ln\frac{1}{4}$

Set $N(t)=\frac{1}{4}N(0)$

$N(0)e^{\frac{1}{10}\ln\frac{1}{4}t} = \frac{1}{2}N(0)$

or $t=\frac{\ln\frac{1}{2}}{\frac{1}{10}\ln\frac{1}{4}} \simeq$ 8.3 years approximately.

Example 4.4 Let N(t) be the population at time t and Let N_0 denote the initial population, that is, $N(0)=N_0$.

Find the solution of the model

$$\frac{dN}{dt} = aN(t) - bN(t)^2$$

with initial condition

$N(0)=N_o$

Solution: This is a separable differential equation, and its solution is

$$\int_0^t \frac{dN(s)}{aN(s) - bN(s)^2} ds = \int_0^t ds = t$$

$$\frac{1}{aN - bN^2} = \frac{1}{N(a - bN)} = \frac{A}{N} + \frac{B}{a - bN}$$

To find A and B, observe that

$$\frac{A}{N} + \frac{B}{a - bN} = \frac{A(a - bN) + BN}{N(a - bN)} = \frac{Aa + (B - bA)N}{N(a - bN)}$$

Therefore, Aa+(B-bA)N=1. Since this equation is true for all values of N, we see that Aa=1 and B-bA=0. Consequently, A=$\frac{1}{a}$, B=b/a, and

$$\int_{N_0}^{N} \frac{ds}{s(a - bs)} = \frac{1}{a} \int_{N_o}^{N} \left(\frac{1}{s} + \frac{b}{a - bs}\right) ds$$

$$= \frac{1}{a}\left[\ln\frac{N}{N_O} + \ln\left|\frac{a - bN_O}{a - bN} - \right|\right]$$

$$\frac{1}{a}\ln\frac{N}{N_o} \left| \frac{a - bN_o}{a - bN} \right|$$

Thus

$$at = \ln\frac{N}{N_0}\left|\frac{a - bN_0}{a - bN}\right|$$

It can be verified that $\frac{a - bN_o}{a - bN(t)}$ is always positive for $0<t<\infty$. Hence

$$at = \ln \frac{N}{N_o}\frac{a-bNo}{a-bN}$$

Taking exponentials of both sides of this equation gives

$$e^{at} = \frac{N}{N_O}\frac{a-bN_O}{a-bN}$$

$$N_0(a\text{-}bN)e^{at} = (a\text{-}bN_0)N$$

Bringing all terms involving N to the left-hand side of this equation, we see that

$$[a\text{-}bN_o + bN_0e^{at}]\, N(t) = aN_0e^{at}$$

$$\text{or } N(t) = \frac{aN_Oe^{at}}{a-bN_O+bN_Oe^{at}}$$

4.3 Radio-active Decay and Carbon Dating

As discussed in Section 1.4.2. a radioactive substance decomposes at a rate proportional to its mass. This rate is called the **decay rate.** If m(t) represents the mass of a substance at any time, then the decay rate $\frac{dm}{dt}$ is proportional to m(t). Let us recall that the **half-life** of a substance is the amount of time for it to decay to one-half of its initial mass.

Example 4.5. A radioactive isotope has an initial mass 200mg, which two years later is 50mg. Find the expression for the amount of the isotope remaining at any time. What is its half-life?

Solution: Let m be the mass of the isotope remaining after t years, and let -k be the constant of proportionality. Then the rate of decomposition is modeled by

$$\frac{dm}{dt} = -\,km,$$

where minus sign indicates that the mass is decreasing. It is a separable equation. Separating the variables, integrating, and adding a constant in the form lnc, we get

$\ln m + \ln c = -kt$

Simplifying,

$\ln mc = -kt$ (4.3)

or $mc = e^{-kt}$

or $m = c_1 e^{-kt}$, where $c_1 = \frac{1}{c}$

To find c_1, recall that m =200 when t=0. Putting these values of m and t in (4.3) we get

$200 = c_1 e^{-ko} = c_1.1$

or $c_1=200$

and $m = 200e^{-kt}$ (4.4)

The value of k may now be determined from (4.4) by substituting t=2, m=150.

$150 = 200 e^{-2k}$

or $e^{-2k} = \frac{3}{4}$

or $-2k = \ln \frac{3}{4}$

This gives

$k = \frac{1}{2}\ln\frac{4}{3} = \frac{1}{2}(0.2877) = 0.1438 \approx 0.14$

The mass of the isotope remaining after t years is then given by

$m(t) = 200e^{-.1438t}$

The half-life t_h is the time corresponding to m=100mg.

Thus

$100 = 200\, e^{-0.14t_h}$

or $\frac{1}{2} = e^{-0.14t_h}$

or $t_h = -\frac{1}{0.14}\ln 0.5 = \frac{-0.693}{-0.14} = 4.95$ years

Carbon Dating: The key to the carbon dating of paintings and other materials such as fossils and rocks lies in the phenomenon of radioactivity discovered at the turn of the century. The physicist Rutherford and his colleagues showed that the atoms of certain radioactive elements are unstable and that within a given time period a fixed portion of the atoms spontaneously disintegrate to form atoms of a new element. Because radioactivity is a property of the atom, Rutherford showed that the radioactivity of a substance is directly proportional to the number of atoms of the substance present. Thus, if N(t) denotes the number of atoms present at time t, then $\frac{dN}{dt}$, the number of atoms that disintegrate per unit time, is proportional to N; that is,

$$\frac{dN}{dt} = -\lambda N \qquad (4.5)$$

The constant λ, which is positive, is known as the decay constant of the substance. The larger λ is, the faster the substance decays.

To compute the half life of substance in terms of λ, assume that at time $t=t_0$, $N(t_0)=N_0$. The solution of the initial value problem

$$\left.\begin{aligned} \frac{dN}{dt} &= -\lambda N \\ N(t_0) &= N_0 \end{aligned}\right\} \qquad (4.6)$$

is

$$N(t)=N_0 e^{-\lambda(t-t_o)}$$

or $$\frac{N}{N_o} e^{-\lambda(t-t_o)}$$

Taking logarithms of both sides we obtain

$$-\lambda\,(t-t_0)=\ln\frac{N}{N_o} \qquad (4.7)$$

If $\frac{N}{N_o} = \frac{1}{2}$, then $-\lambda\,(t-t_0)=\ln\frac{1}{2}$, so that

$$t-t_0 = \frac{\ln 2}{\lambda} = \frac{0.6931}{\lambda}$$

Thus the half life of a substance is ln2 divided by the decay constant λ**.**

The half-life of many substances have been determined and are well published. For example, half-life of carbon-14 is 5568 years, and the half-life of uranium 238 is 4.5 billion years.

Remark 4.3.1 a) In (4.5) λ is positive and is decay constant. We may write equation (4.5) in the form

$$\frac{dN}{dt} = \lambda N,$$ where λ is negative constant, that is, $\lambda < 0$.

b) The dimension of λ is reciprocal time. It t is measured in years, then λ has the dimension of reciprocal years, and if t is measured in minutes, then λ has the dimension of reciprocal minutes.

c) From (4.7) we can solve for

$$t\text{-}t_0 = \frac{1}{\lambda}\ln\frac{N_o}{N} \qquad (4.8)$$

If t_0 is the time the substance was initially formed or manufactured, then the age of the substance is $\frac{1}{\lambda}\ln\frac{N_0}{N}$. The decay constant λ is known or can be computed in most cases. N can be computed quite usually. Computation or pre-knowledge of N_0 will yield the age of the substance.

By the Libby's discovery discussed in Section 1.4.2. the present rate R(t) of disintegration of the C-14 in the sample is given by $R(t) = \lambda N(t) = \lambda N_0 e^{-\lambda t}$ and the original rate of disintegration is $R(o) = \lambda N_0$. Thus

$$\frac{R(t)}{R(0)} = e^{-\lambda t} \quad \text{so that}$$

$$t = \frac{1}{\lambda}\ln\frac{R(o)}{R(t)} \qquad (4.9)$$

d) If we measure R(t), that present rate of disintegration of the C-14 in the charcoal and observe that R(o) must equal the rate of disintegration of the C-14 in the comparable amount of living wood then we can compute the age t of the charcoal.

e) The process of estimating the age of an artifact is called **carbon dating**.

Example 4.6 : Suppose that we have an artifact, say a piece of fossilized wood, and measurements show that the ratio of C-14 to carbon in the sample is 37% of the current ratio. Let us assume that the wood died at time 0, then compute the time T it would take for one gram of the radio active carbon to decay this amount.

Solution: By model (1.10)

$$\frac{dm}{dt} = km$$

This is a separable differential equation. Write it in the form

$$\frac{1}{m} dm = kdt$$

Integrate it to obtain

$$\ln|m| = kt + c$$

Since mass is positive, lml=m and

$\ln(m)=kt+c$.

Then

$m(t) = e^{kt+c} = Ae^{kt}$, where $A = e^c$ is positive constant. Let at some time, designated at time zero, there are M grams present. This is called the initial mass. Then

$m(o) = A = M$, so

$m(t) = Me^{kt.}$

If at some later time T we find that there are M_T grams, then

$m(T) = M_T = Me^{kT}$.

Then

$$\ln\left(\frac{M_T}{M}\right) = kT$$

hence

$$k = \frac{1}{T}\ln\left(\frac{M_T}{M}\right)$$

This gives us k and determines the mass at any time:

$m(t) = Me^{\frac{t}{T}\ln\left(\frac{M_T}{M}\right)}$

Let T=λ be the time at which half of the mass has radiated away, that is, half-life. At this time, half of the mass remains, so M_T=M/2 and M_T/M = $\frac{1}{2}$.

Now the expression for mass becomes

$m(t) = Me^{\frac{t}{\lambda}\ln(\frac{1}{2})}$

or $m(t) = Me^{-\frac{t}{\lambda}\ln 2}$

Half-life of C-14 is 5600 years approximately, that is,

λ = 5600

$\frac{\ln 2}{5600} \approx -0.00012378$

$\approx$ means approximately equal (all decimal places are not listed).

Therefore

$m(t)=Me^{-0.00012378t}$

or $\frac{m(t)}{M} = 0.37 = e^{-0.00012378\,t}$

by the given condition that $\frac{m(t)}{M}$ is .37 during t.

$\therefore$ T= - $\frac{\ln(0.37)}{0.00012378} =$ 8031 years approximately.

Example 4.7 (a) A fossilized bone is found to contain one thousandth the original amount of C-14. Determine the age of fossil.

(b) Use the information provided in part (a) to determine the approximate age of a piece of wood found in an archaeological excavation at the site to date prehistoric paintings and drawing on the walls and ceilings of a cave in Lascaux, France, provided 85.5% of the C-14 had decayed.

Solution: a. The separable differential equation

$\frac{dN}{dt} = k\,N(t)$, where k is the constant of proportionality of decay, models the phenomenon as discussed above.

The solution is

$N(t) = N_0e^{kt}$ (say λ = -k, if we want to put in the form of the above discussion).

Half-life of C-14 is approximately 5600 years

$$\frac{N_o}{2} = N(5600)$$

or $\frac{1}{2}$ $N_0 = N_0e^{5600k}$. By cancelling N_0 and taking logarithm of both sides we get

$$5600\ k = \ln\frac{1}{2} = -\ln 2$$

$$\text{or } k = -\frac{\ln 2}{5600} = -0.00012378$$

Therefore

$$N(t) = N_0\, e^{-0.00012378t}$$

With $N(t) = \frac{1}{1000}N_0$ we have

$$\frac{1}{1000}N_0 = N_0e^{-0.00012378t}$$

$$-0.00012378t = \ln\frac{1}{1000} = -\ln 1000.$$ Thus

$$t = \frac{\ln 1000}{0.00012378} \approx 55800 \text{ yrs}$$

(b) Let $N(t)= N_0e^{kt}$ where k= -0.00012378 by part (a).

85.5% of C-14 had decayed; that is,

$N(t) = 0.145\, N_0$

or $N_0e^{-0.00012378t} = 0.145\, N_0$

Taking logarithm of both sides and solving for t, we get

$t \approx 15{,}600$ years

4.4 Mixture of Two Salt Solutions

Example. 4.8 A tank contains 300 litres of fluid in which 20 grams of salt is dissolved. Brine containing 1 gm of salt per litre is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate. Find the number N(t) of grams of salt in the tank at time t.

Solution: By the data given in this example we have

$P(t) = N(t)$

$n=4,\ p=1,\ m=300$

in the model (1.23)

$$\frac{dP(t)}{dt} = 4 - \frac{P}{300}.4 = 4 - \frac{P}{75}$$

or $$\frac{dP(t)}{dt} + \frac{P}{75} = 4$$

This is a linear differential of first order in P whose integrating factor is

$$e^{\int \frac{1}{75} dt} = e^{\frac{1}{75}t}$$ (See Section 2.3)

Solution is given by

$$P(t).e^{\frac{1}{75}t} = \int 4e^{\frac{1}{75}t} dt + c$$

$$\text{P(t) = 300 + c.}e^{-\frac{1}{75}t}$$

Since P(0) = 20 is given we get

20=P(0) = 300+Ce^{0}, that is c= -280

Thus P(t) = 300 - 280$e^{-\frac{1}{75}t}$

4.5 Series Circuits

Let a series circuit contain only a resistor and an inductor as shown in Figure 4.2

Figure 4.2 LR Series circuit

By Kirchhoff's second law the sum of the voltage drop across the inductor $\left(\mathrm{I}\frac{\mathrm{di}}{\mathrm{dt}}\right)$ and the voltage drop across the resistor (iR) is the same as the impressed voltage (E(t)) on the circuit. Current at time t, i(t), is the solution of the differential equation.

$$\mathrm{I}\frac{di}{dt} + Ri = E(t) \qquad (4.10)$$

where I and R are constants known as the inductance and the resistance respectively.

The voltage drop across a capacitor with capacitance C is given by $\frac{q(t)}{C}$, where q is the charge on the capacitor. Hence, for the series circuit shown in Figure 4.3 we get the following equation by applying Kirchhoff's second law

$$Ri + \frac{1}{C}q = E(t) \qquad (4.11)$$

Figure 4.3 RC Series Circuit

Since $i = \frac{dq}{dt}$, (4.11) can be written as

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t) \qquad (4.12)$$

Example 4.9 Find the current in a series RL circuit in which the resistance, inductance, and voltage are constant. Assume that i(o)=0; that is initial current is zero.

Solution:_ It is modeled by (4.10)

$$I\frac{di}{dt} + Ri = E(t)$$

$$\text{or} \quad \frac{di}{dt} + \frac{R}{I}i = \frac{E(t)}{I} \qquad (4.13)$$

Since I, R and E are constant

(4.13) is linear equation of first-order in i with integrating factor

$$e^{\int \frac{R\,dt}{\mathrm{I}}} = e^{\frac{R\,t}{\mathrm{I}}}$$

The solution of (4.13) is

$$i(t)e^{\frac{R\,t}{\mathrm{I}}} = \int \frac{E}{\mathrm{I}} e^{\frac{R\,t}{\mathrm{I}}}\,dt$$

$$\text{or} \quad i(t)e^{\frac{R}{\mathrm{I}}t} = \frac{E}{\mathrm{I}} e^{\frac{R\,t}{\mathrm{I}}} \frac{\mathrm{I}}{R} + c$$

$$\text{or } i(t) = \frac{E}{R} + ce^{-\frac{R}{\mathrm{I}}t} \qquad (4.14)$$

Since i(0) = 0, c = $-\frac{E}{R}$

Putting this value of c in (4.14) we get

$$i(t) = \frac{E}{R}\,(1 - e^{-\frac{R}{\mathrm{I}}t})$$

Example 4.10 A 100-volt electromotive force is applied to an RC series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farads. Find the charge q(t) on the capacitor if q (0)=0. Find the current i(t).

Solution: The phenomenon is modeled by (4.12):

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t) \quad , \quad \text{where}$$

R=200, C=10^{-4}, E(t) = 100

Thus

$$\frac{dq}{dt} + \frac{1}{200}10^4 q = \frac{1}{2} \qquad (4.15)$$

This is a linear differential equation of first-order

The integrating factor is $e^{\int 50dt} = e^{50t}$

and so the solution of (4.15) is

$$q(t)e^{50t}=\frac{1}{2}\int e^{50t}dt + c$$

$$\text{or } q(t)=\frac{1}{100}+ce^{-50t}$$

$$q(0)=0=\frac{1}{100}+ce^{-50.0}$$

or $c = -\frac{1}{100}$ and so

$$q(t) = \frac{1}{100} - \frac{1}{100}e^{-50t}$$

$$\frac{dq(t)}{dt} = \frac{1}{2}e^{-50t}$$

But $i = \frac{dq(t)}{dt}$ and so

$$i=\frac{1}{2}e^{-50t}$$

4.6 Survivability with AIDS

Equation (1.31) provides survival fraction S(t). It is a separable equation and its solution is

$S(t) = S_i+(1-S_i)e^{-kt}$:

Given equation is

$$\frac{dS(t)}{dt} = -k(S(t)-S_i)$$

$$\frac{dS}{S(t)-S_i} = -kdt$$

Integrating both sides, we get

$\ln|S(t)-S_i|=-kt+\ln c$

$$\ln\frac{|S(t)-S_i|}{c} = -kt$$

or $\dfrac{S(t)-S_i}{c} = e^{-kt}$

$$S(t)=S_i+ce^{-kt}$$

Let S(0)=1 then c=1-Si. Therefore

$S(t) = S_i + (1-S_i)e^{-kt}$

We can rewrite this equation in the equivalent form.

$S(t)=S_i+(1-S_i)e^{-t/T}$

where, in analogy to radioactive nuclear decay,

T is the time required for half of the mortal part of the cohort to die-that is, the survival half life.

Example 4.11 Consider the initial-value problem

$$\frac{dS(t)}{dt} = -k(S(t)-S_i) \qquad (4.16)$$
$$S(0)=1$$

as the survivability with AIDS.

(a) Show that, in general, the half-life T for the mortal part of the cohort to die is $T = \dfrac{\ln 2}{k}$

(b) (b) Show that the solution of the initial value problem can be written as

$S(t)=S_i+(1-S_i)2^{-t/T}$ (4.17)

Solution: The solution of the separable differential equation in (4.16) is

$S(t) = (1-S_i)e^{-kt}+S_i$ (4.18)

Let S(t) = $\frac{1}{2}$ S(0), and solving for t we obtain the half=life T = $\frac{\ln 2}{k}$

(b) Putting $k = \frac{T}{\ln 2}$ in (4.18) we obtain

$$S(t) = Si + (1 - Si)e^{-\frac{T}{\ln 2}t}$$

4.7 Draining a Tank

In Section 1.4.8 modeling of draining a tank is discussed. Equation (1.26) models the rate at which the water level is dropping.

Example 4.12 A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. Find the height h of water in the tank at any time t if the initial height of the water is H.

Solution: As discussed in Section 1.4.8, h(t) is the solution of the equation (1.26); that is,

$$\frac{dh}{dt} = -\frac{B}{A}\sqrt{2gh} \qquad (4.19)$$

where A is the cross section area of the cylinder and B is the cross sectional area of the orifice at the base of the container.

(4.19) can be written as

$$\frac{dh}{\sqrt{h}} = -\frac{B}{A}\sqrt{2g} \;\; dt$$

or $\frac{dh}{\sqrt{h}} = Cdt$ where

$C = -\frac{B}{A}\sqrt{2g}$

By integrating this equation we get

$2h^{\frac{1}{2}} = Ct + c'$

For t=0 h=H and so

$c' = 2H^{\frac{1}{2}}$ Therefore

$$h(t) = \left(\frac{Ct + 2H^{\frac{1}{2}}}{2} \right)^2$$

4.8 Economics and Finance

We have presented models of supply, demand and compounding interest in Section 1.4.3. We solve those models, namely equations (1.11) and (1.16).

(1.11), that is equation

$\frac{dP}{dt} = k(D - S)$ is a separable differential equation of first-order. We can write it as

dP=k(D-S) dt.

Integrating both sides, we get

P(t)=k(D-S)t+A

where A is a constant of integration.

Solution of (1.16), which is also a separable equation, is

$$S(t)=S(0)\, e^{rt} \qquad (4.20)$$

where S(0) is the initial money in the account

Example 4.13 Find solution of the model of Example 1.21 with no initial demand (D(0)=0).

Solution: The model is

$$\frac{dD}{dt}=k\frac{t}{\sqrt{D}}$$

This can be written as

$$D^{1/2}dD=k\,tdt$$

Integrating both sides we get

$$\frac{2}{3}D^{\frac{3}{2}}=k\frac{1}{2}t^{2}+A,$$

where A is a constant integration. If Demand D=0 at the initial time t=0, then A=0 and demand D(t) at any time t is given by

$$D(t)=\left(\frac{3kt^{2}}{4}\right)^{\frac{2}{3}}$$

4.9 Mathematics Police Women

The time of death of a murdered person can be determined with the help of modeling through differential equation. A police personnel discovers the body of a dead person presumably murdered and the problem is to estimate the time of death. The body is located in a room that is kept at a constant 70 degree F. For some time after the death, the body will radiate heat into the cooler room, causing the body's temperature to decrease assuming that the victim's temperature was normal 98.6F at the time of death. Forensic expert will try to estimate this time from body's current temperature and calculating how long it would have had to lose heat to reach this point.

According to Newton's law of cooling, the body will radiate heat energy into the room at a rate proportional to the difference in temperature between the body and the room. If T(t) is the body temperature at time t, then for some constant of proportionality k,

$T'(t)=k[T(t)-70]$

This is a separable differential equation and is written as

$$\frac{1}{T-70}dT=kdt$$

Upon integrating both sides, one gets

$\ln|T-70|=kt+c$

Taking exponential, one gets

$|T-70|=e^{kt+C}=Ae^{kt}$

where $A = e^C$. Then

$T-70= \pm Ae^{kt}= Be^{kt}$

Then

$T(t)=70 + Be^{kt}$

Constants k and B can be determined provided the following information is available: Time of arrival of the police personnel, the temperature of the body just after his arrival, temperature of the body after certain interval of time.

Let the officer arrived at 10.40 p.m. and the body temperature was 94.4 degrees. This means that if the officer considers 10:40 p.m. as t=0 then

T(0)=94.4=70+B and so

B=24.4 giving

$T(t)=70 + 24.4\, e^{kt.}$

Let the officer makes another measurement of the temperature say after 90 minutes, that is, at 12.10 a.m. and temperature was 89 degrees. This means that

$T(90)=89=70+24.4\, e^{90k}$

Then

$$e^{90k} = \frac{19}{24.4},$$

so

$$90k = \ln\left(\frac{19}{24.4}\right)$$

and

$$k = \frac{1}{90}\ln\left(\frac{19}{24.4}\right)$$

The officer has now temperature function

$$T(t) = 70 + 24.4\,e^{\frac{t}{90}\ln\left(\frac{19}{24.4}\right)}$$

In order to find when the last time the body was 98.6 (presumably the time of death), one has to solve for time the equation

$$T(t) = 98.6 = 70 + 24.4\,e^{\frac{t}{90}\ln\left(\frac{19}{24.4}\right)}$$

To do this, the officer writes

$$\frac{28.6}{24.4} = e^{\frac{t}{90}\ln\left(\frac{19}{24.4}\right)}$$

and takes logarithms of both sides to obtain

$$\ln\left(\frac{28.6}{24.4}\right) = \frac{t}{90}\ln\left(\frac{19}{24.4}\right)$$

Therefore, the time of death, according to this mathematical model, was

$t = \frac{90\ln(28.6/24.4)}{\ln(19/24.4)}$ which is approximately –57.0.7 minutes.

The death occurred approximately 57.07 minutes before the first measurement at 10.40 p.m. , that is at 9.43 p.m. approximately

4.10 Drug Distribution (Concentration) in Human Body

To combat the infection to human a body appropriate dose of medicine is essential. Because the amount of the drug in the human body decreases with time medicine must be given in multiple doses. The rate at which the level y of the drug in a patient's blood decays can be modeled by the decay equation

$$\frac{dy}{dt} = -ky$$

where k is a constant to be experimentally determined for each drug. If initially, that is, at t=0 a patient is given an initial dose y_p, then the drug level y at any time t is the solution of the above differential equations, that is,

$y(t)=y_p\ e^{-kt}$

Remark: 4.10.1. In this model it is assumed that the ingested drug is absorbed immediately which is not usually the case. However, the time of absorption is small compared with the time between doses.

Example 4.14: A representative of a pharmaceutical company recommends that a new drug of his company be given every T hours in doses of quantity y_0, for an extended period of time. Find the steady state drug in the patient's body.

Solution: Since the initial dose is y_0, the drug concentration at any time t ≥o is found by the equation $y=y_0e^{-kt}$, the solution of the equation $\frac{dy}{dt}=-ky$

At t=T the second dose of y_0 is taken, which increases the drug level to

$y(T)=y_0+y_0\, e^{-kT} = y_0(1+e^{-kT})$

The drug level immediately begins to decay. To find its mathematical expression we solve the initial-value problem:

$$\frac{dy}{dt}=-ky$$

$y(T)=y_0(1+e^{-kT})$

Solving this initial value problem we get

$y=y_0(1+e^{-kT})e^{-k(t-T)}$

This equation gives the drug level for t>T. The third dose of y_0 is to be taken at t=2T and the drug just before this dose is taken is given by

$$y = y_o\left(1+e^{-kT}\right)e^{-k(2T-T)} = y_o(1+e^{-kT})e^{-kT}$$

The dosage y_0 taken at t=2T raises the drug level to

$y(2T) = y_0 + y_0(1+e^{-kT})e^{-kT} = y_0(1+e^{-kT}+e^{-2kt})$

Continuing in this way, we find after (n+1)th dose is taken that the drug level is

$y(nT)=y_0(1+e^{-kT}+e^{-2kT}+\ldots..+e^{-nkT})$

We notice that the drug level after $(n+1)^{th}$ dose is the sum of the first n terms of a geometric series, with first term as y_o and the common ratio e^{-kT}. This sum can be written as

$$y(nT) = \frac{y_o(1-e^{-(n+1)kT})}{1-e^{-kT}}$$

As n becomes large, the drug level approaches a steady state value, say y_s given by

$$y_s = \lim_{n\to\infty} y(nT)$$

$$= \frac{y_o}{1-e^{-kT}}$$

The steady state value y_s is called the saturation level of the drug.

4.11 A Pursuit Problem

Figure 4.4

A dog chasing a rabbit is shown in Figure 4.4. The rabbit starts at the position (0,0) and runs at a constant speed v_R along the y-axis. The dog starts chase at the position (1.0) and runs at a constant speed v_D so that its line of sight is always directed at the rabbit. If $v_D > v_R$, the dog will catch the rabbit; otherwise the rabbit gets away. Finding the function representing the pursuit curve gives the path the dog follows. Since the dog always runs directly at the rabbit during the pursuit, the slope of the line of sight between the dog and the rabbit at any time t is given by

$$m = \frac{y - y_R}{x - x_R} = \frac{y - y_R}{x}$$

If we assume that the line of sight is tangent to the pursuit curve y=f(x), then $m = \frac{dy}{dx}$ and therefore $\frac{dy}{dx} = \frac{y - y_R}{x}$ (4.21)

is the mathematical model of the "Pursuit Problem". The solution of (4.21) will give the path taken by the dog.

The position of the dog at any time t>0 is (x,y), and the y coordinate of the rabbit at the corresponding time is $y_R = 0 + v_R t = v_R t$, so

$$\frac{dy}{dx} = \frac{y - v_R t}{x}$$

or $$x\frac{dy}{dx} = y - v_R t$$

Implicitly differentiating this expression with respect to x yields

$$xy'' + y' = y' - v_R\left(\frac{dt}{dx}\right)$$

where $y'' = \left(\frac{d^2y}{dx^2}\right)$, $y' = \frac{dy}{dx}$

This may be written as

$$\frac{xy''}{v_R} = -\frac{dt}{dx} \qquad (4.22)$$

Finally, we note that the speed of the dog can be written as

$$v_D = \frac{ds}{dt} = -\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \frac{dx}{dt}$$

Solving this for $\frac{dt}{dx}$, we have

$$\frac{dt}{dx} = -\frac{1}{v_D}\sqrt{1 + (y')^2}$$

Substituting this result into Equation (4.22) yields

$$\frac{xy''}{v_R} = \frac{1}{v_D}\sqrt{1 + (y')^2}$$

Put y'=w, then this equation takes the form

$$\frac{xw'}{v_R} = \frac{1}{v_D} \sqrt{1+w^2}$$

or $$\frac{dw}{\sqrt{1+w^2}} = \frac{v_R}{v_D} \frac{dx}{x}$$

Integrating both sides we get w and by integrating w we get y. The constant of integration can be found by using the initial conditions y(1)=0 and y' (1)=0.

4.12 Harvesting of Renewable Natural Resources

There are many renewable natural resources that humans desire to use. Examples are fishes in rivers and sea and trees from our forests. It is desirable that a policy be developed that will allow a maximal harvest of a renewable natural resource and yet not deplete that resource below a sustainable level. We introduce a mathematical model providing some insights into the management of renewable resources.

Let P(t) denote the size of a population at time t, the model for exponential growth begins with the assumption that $\frac{dP}{dt} = kP$ for some k>0. In this model the relative or specific, growth rate defined by

$$\frac{dP}{dt} / P$$

is assumed to be a constant.

In many cases $\frac{dP}{dt} / P$ is not constant but a function of P, let

$$\frac{dP}{dt} / P = f(P)$$

or $\frac{dP}{dt} = P\ f(P)$

Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its population. The quantity is called the carrying capacity of the environment.

Special cases: (i) f (P)=c_1P +c_2

(ii) If f(0)=r and f(K)=0 then

c_2=r and c_1= -$\frac{r}{k}$, and so (i) takes the form

f (P) = r-($\frac{r}{k}$)P.

Simple Renewable natural resources model is

$$\frac{dP}{dt} = P(r - \frac{r}{K}P)$$

This equation can also be written as

$$\frac{dP}{dt} = P(a - bP)$$

Example 4.15: Find the solution of the following harvesting model

$$\frac{dP}{dt} = P(5 - P) - 4$$

P(o)=P_o

Solution: 4.15 The differential equation can be written as

$$\frac{dP}{dt} = -(P^2 - 5P + 4) = -(P - 4)(P - 1)$$

or $\frac{dP}{(P-4)(P-1)} = -dt$

or $\left(\frac{\frac{1}{3}}{P-4} - \frac{\frac{1}{3}}{P-1} \right) dP = -dt$

Integrating we get

$$\frac{1}{3}\ln\left|\frac{P-4}{P-1}\right| = -t + c$$

or $\frac{P-4}{P-1} = c_1 e^{-3t}$

Setting t=0 and $P=P_0$ we find $c_1=(P_o-4)/(P_o-1)$.

Solving for P we get

$$P(t) = \frac{4(Po-1)-(Po-4)e^{-3t}}{(Po-1)-(Po-4)e^{-3t}}$$

4.13 Exercises

Newton's Law of Cooling/Warming

1. A thermometer reading 100^0 F is placed in a pan of oil maintained at 10^0 F. What is the temperature of the thermometer when t=20 sec, if its temperature is 60^0 F when t = 8 sec?
2. A thermometer is removed from a room where the air temperature is 60^0 F and is taken outside, where the temperature is 10^0 F. After 1 minute the thermometer reads 50^0 F. What is the reading of the thermometer at t=2 minutes? How long will it take for the thermometer to reach 20^0 F .
3. Water is heated to a boiling point temperature 120^0C. It is then removed from the burner and kept in a room of 30^0C temperature.

Assuming that there is no change in the temperature of the room and the temperature of the hot water is 110°C after 3 minutes. (a) Find the temperature of water after 6 minutes (b) Find the duration in which water will cool down to the room temperature?

Population Growth and decay

4. A culture initially has P_o number of bacteria. At t=1 hour, the number of bacteria is measured to be $\frac{3}{2}$ P_0. If the rate of growth is proportional to the number of bacteria P(t) present at time t, determine the time necessary for the number of bacteria to triple.

5. Solve the logistic differential equation:

$$\frac{dN}{dt} = r_0(1 - \frac{N}{k})N,$$

$$t \geq 0,\ N(0) = N_0$$

6. Insects in a tank increase at a rate proportional to the number present. If the number increases from 50,000 to 100,000 in one hour, how many insects are present at the end of two hours.

7. It was estimated that the earth's human population in 1961 was 3,060,000,000. Assuming the population increases at the rate of 2 percent, find the earth's population in 1996 using model of population growth (1.8). Check this number with the actual population of the earth available from authentic sources.

Radio-Active Decay and Carbon Dating

8. A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 30 years it is determined that 0.022% of the initial amount N_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration in proportional to the amount remaining.

9. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 4 hours. If 1 gram of lead is present initially, how long will it take for 80% of the lead to decay?

10. Solve the model obtained in Exercise 32 of chapter 1.

11. In the 1950 excavation at Nippur, a city of Babylonia, charcoal from a roof beam gave a count of 4.09 dis/min/g. Living wood gave 6.68 disintegrations. Assuming that this charcoal was formed during the time of Hammurabi's reign, find an estimate for the likely time of Hammurabi's succession.

Mixture of Two Salt Solutions

12. A tank with a capacity of 600 litres initially contains 200 litres of pure water. A salt solution containing 3 Kg of salt per litre is allowed to run into the tank at a rate of 16 lit/min, and the mixture is then removed at a rate of 12 lit/min. Find the expression for the number of Kilograms of salt in the tank at any time t.

13. A large tank is filled with 600 liters of pure water. Brine containing 2 Kg of salt per litre is pumped into the tank at a rate of 5 litre/min. The well-mixed solution is pumped out at the same rate. Find the number P(t) of kilograms of salt in the tank at time t. What is the concentration of the solution in the tank at t=10 min?

14. A 250-litre tank contains 100 litres of pure water. Brine containing 4 kg of salt per litre flows into the tank at 5 lit/hr. If the well-stirred mixture flows out at 3 lit/hr, find the concentration of salt in the tank at the instant it is filled to the top.

Series circuit

15. A series RL circuit has a resistance 20 ohms, and an inductance of 1 henry, and an impressed voltage of 12 volts. Find the current i(t) if the initial current is zero.

16. An electromotive force

$$E(t)=\begin{cases} 120, & 0 \leq t \leq 20 \\ 0\ , & t > 20 \end{cases}$$

is applied to an LR series circuit in which the inductance is 20 henries and the resistance is 2 ohms. Find the current i(t) if i(0)=0

Survivability with AIDS

17. Find survival fraction S(t) with aids after 2 years by applying model (1.31).

Draining a Tank

18. A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. Let us assume that the height of the tank is 10 ft. high and has radius 2 ft. and circular hole has radius ½ inches. If the tank is initially full, how long it will take to empty?

Economics and Finance

19. What rate of interest payable annually is equivalent to 6% continuously compounded?

20. Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later? How much he has in the account if the interest were compounded monthly.

Drug distribution(Concentration) in Human Body

21. A drug with k=0.01 is administered every 12 hours in doses of 4 mg. Calculate the amount of the drug in the patient's body after the 4^{th} dose is taken.

22. A drug with k=0 .030 is to be administered in doses of y_0=4mg, gradually building upto a saturation level in the patient of y_s=20mg. Calculate the dosage time interval. Assume that the time interval is in hours.

Pursuit Problem

23. Complete the solution of the pursuit problem.

24. Solve the pursuit problem if $\nu_R=3$ and $\nu_D=2$. Draw the path pursued by the dog.

Harvesting of Renewable Natural Resources

25(a) Solve the initial value problem

$$\frac{dP}{dt} = P(5-P) - \frac{25}{4}$$

$P(0)=P_0$

25 (b) Find when the population becomes extinct in the case $0< P_0<5/2$