SAMPLE LESSON: MATHEMATICS

Class: Upper Sixth Mathematics

Module: Plane Geometry and Solid Figures	**TOPIC:** Complex Numbers
Title of Lesson: Polar form (trigonometric form) and exponential form of a non-zero complex number	**Duration of Lesson:** 120mins

Name of Authors: Inspectorate of Pedagogy/Sciences for the Far North Region

Module 4 : **PLANE GEOMETRY AND SOLID FIGURES**

Topic: Complex Numbers

Lesson: Polar form (trigonometric form) and exponential form of a non-zero complex number

Objectives: At the end of this lesson, the learners should be able to:

1) Calculate the modulus of a complex number
2) Find Argument of a non-zero complex number;
3) Give the polar form of a non-zero complex number;
4) Give the exponential form of a non-zero complex number.

Key question:

What is the polar form of a non-zero complex number?

Prerequisite knowledge:

- ✓ Verify whether students can calculate the norm of a vector.
- ✓ Verify whether students can determine the measurement of the angle between the positive X-axis and a vector.

Motivation: The study of complex numbers comes to reinforce our knowledge and skills necessary to study plane geometry.

Didactic materials

Chalk, colour chalk, Chalkboard, ruler and set square.

REFERENCES

- EWANE ROLAND ALUNGE. Advanced Level Pure Mathematics Made Easy First Edition.
-Pure Mathematics With Mechanics Teaching Syllabuses (January 2020)

Stages / Duration	Teaching /Learning activities	Teacher's Activities	Learners' Activities	Learning Points	Observations
Introduction (5mins)	**<u>Verification of Pre-requisites</u>** **<u>Exercise</u>** a) Calculate the norm of $\overrightarrow{OA}\binom{0}{4}$ given in the Cartesian plane. b) Determine the measure of the angle between $\vec{\imath}$ and $\overrightarrow{OA}$.	-Copies questions on the board -Calls students to the board	-Solve on the board as called by the teacher	**<u>Verification of Pre-requisites</u>** **<u>Exercise</u>** a) Calculate the norm of $\overrightarrow{OA}\binom{0}{4}$ given in the Cartesian plane. b) Determine the measure of the angle between $\vec{\imath}$ and $\overrightarrow{OA}$.	
Lesson Development and Summary (100mins)	**<u>Activity</u>** In an orthonormal reference system (O, $\vec{\imath}$, $\vec{\jmath}$),we consider the points A,B and C with respective affixes $z_A = 2, z_B = 3i$ and $z_C = 2 + 2i$ 1) Plot the points A, B and C in this reference system. 2) Calculate the values of OA,OB and OC. 3) Determine the measure of the following angles: a) The angle between $\vec{\imath}$ and $\overrightarrow{OA}$. b) The angle between $\vec{\imath}$ and $\overrightarrow{OB}$. c) The angle between $\vec{\imath}$ and $\overrightarrow{OC}$.	-Copies activity on the chalkboard -Instructs students to copy in their notebooks and allows them 10 minutes to research	-Follow the instructions and carry out the activity while interacting with each other	**<u>Activity</u>** In an orthonormal reference system (O, $\vec{\imath}$, $\vec{\jmath}$),we consider the points A,B and C with respective affixes $z_A = 2, z_B = 3i$ and $z_C = 2 + 2i$ 1) Plot the points A, B and C in this reference system. 2) Calculate the values of OA,OB and OC. 3) Determine the measure of the following angles: a) The angle between $\vec{\imath}$ and $\overrightarrow{OA}$. b) The angle between $\vec{\imath}$ and $\overrightarrow{OB}$. c) The angle between $\vec{\imath}$ and $\overrightarrow{OC}$.	

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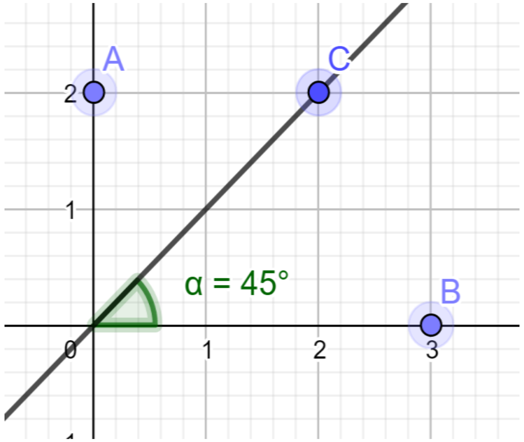
<u>Resolution</u>

1)



2)OA=$\sqrt{0^2+2^2}$=2,OB$\sqrt{3^2+0^2}$=3

OC=$\sqrt{2^2+2^2}$=$\sqrt{8}$

3)According to the figure above,

a)The angle between $\vec{\imath}$ and $\overrightarrow{OA}$ is $\frac{\pi}{2}$

b)The angle between $\vec{\imath}$ and $\overrightarrow{OB}$ is 0

c) The angle between $\vec{\imath}$ and $\overrightarrow{OC}$ is $\frac{\pi}{4}$.

<u>Observation</u>

For example

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-Copies notes on the board

-Explains concepts

-Dictates notes

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-Copy notes in their books

Copy notes in their notebooks

Follow up

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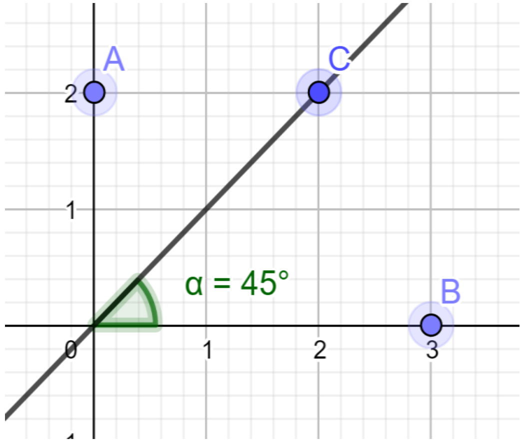
<u>Resolution</u>

1)



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	→) $\sqrt{8}$ Is called modulus of the complex number $z = 2 + 2i$. →) ($\vec{i}$;$\overrightarrow{OC}$)= $\frac{\pi}{4}$ will be called **argument** of the complex number $z = 2 + 2i$. **<u>Definition</u> (modulus)** Let $z = a + bi$ be a complex number, We call the modulus of z the **strictly positive** real number denoted by $\lvert z\rvert$ and define by $\lvert z\rvert=\sqrt{a^2 + b^2}$ **<u>Example</u>** Calculate the modulus of the complex number $z = 2 + 3i$ **<u>Resolution</u>** $\lvert z\rvert=\sqrt{2^2 + 3^2} = \sqrt{13}$ **<u>Properties</u>** Let z and z' two complex numbers →) $\lvert z\rvert=0$ if and only if $z = 0$. →) $\lvert z \times z'\rvert=\lvert z\rvert \times \lvert z'\rvert$ →) $\lvert z^n\rvert = \lvert z\rvert^n \; \forall n \in \mathbb{N}$ →) $\lvert \bar{z}\rvert=\lvert z\rvert$ →) $\left\lvert \frac{z}{z'} \right\rvert = \frac{\lvert z\rvert}{\lvert z'\rvert} \; \forall z' \in \mathbb{C}^*$ **<u>Remark</u>** For any $M(z)$ and $M'(z')$ $\boldsymbol{MM' = \lvert Z_{M'} - Z_M\rvert}$ **<u>Definition</u> (argument)**	Solves the examples one after the other while questioning the students	as the teacher explains, asking and answering questions where necessary	→) $\sqrt{8}$ Is called modulus of the complex number $z = 2 + 2i$. →) ($\vec{i}$;$\overrightarrow{OC}$)= $\frac{\pi}{4}$ will be called **argument** of the complex number $z = 2 + 2i$. **<u>Definition</u> (modulus)** Let $z = a + bi$ be a complex number, We call the modulus of z the **strictly positive** real number denoted by $\lvert z\rvert$ and define by $\lvert z\rvert=\sqrt{a^2 + b^2}$ **<u>Example</u>** Calculate the modulus of the complex number $z = 2 + 3i$ **<u>Resolution</u>** $\lvert z\rvert=\sqrt{2^2 + 3^2} = \sqrt{13}$ **<u>Properties</u>** Let z and z' two complex numbers →) $\lvert z\rvert=0$ if and only if $z = 0$. →) $\lvert z \times z'\rvert=\lvert z\rvert \times \lvert z'\rvert$ →) $\lvert z^n\rvert = \lvert z\rvert^n \; \forall n \in \mathbb{N}$ →) $\lvert \bar{z}\rvert=\lvert z\rvert$ →) $\left\lvert \frac{z}{z'} \right\rvert = \frac{\lvert z\rvert}{\lvert z'\rvert} \; \forall z' \in \mathbb{C}^*$ **<u>Remark</u>** For any $M(z)$ and $M'(z')$ $\boldsymbol{MM' = \lvert Z_{M'} - Z_M\rvert}$ **<u>Definition</u> (argument)**	

Stages / Duration	Teaching /Learning activities	Teacher's Activities	Learners' Activities	Learning Points	Observations
	Let z be a non-zero complex number, M the image of the complex number z ,in the complex plane with an orthonormal reference system (O, $\vec{\imath}$, $\vec{\jmath}$), -We call **Argument** of z the **angle** $\left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ denoted $\boldsymbol{Arg(z)}$ - We call **argument** of z the **measure** of **angle** $\left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ denoted $\boldsymbol{arg(z)}$ $\mathbf{Arg(z)} = \left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ $\mathbf{arg}(z) = \mathbf{mes}\left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ <u>**Example**</u> $\mathbf{Arg}(2+2i) = \left(\widehat{\frac{\pi}{4}}\right)$ $\mathbf{arg}(2+2i) = \frac{\pi}{4}$ <u>**Remark**</u> If the affix of the vector $\vec{u}$ is $Z_{\vec{u}}$,then $\mathbf{arg}(z_{\vec{u}}) = \mathbf{mes}(\widehat{\vec{\imath};\vec{u}})$ <u>**Polar form of a non-zero complex number**</u> Let $z = a + ib$ be a non-zero complex number			Let z be a non-zero complex number, M the image of the complex number z ,in the complex plane with an orthonormal reference system (O, $\vec{\imath}$, $\vec{\jmath}$), -We call **Argument** of z the **angle** $\left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ denoted $\boldsymbol{Arg(z)}$ - We call **argument** of z the **measure** of **angle** $\left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ denoted $\boldsymbol{arg(z)}$ $\mathbf{Arg(z)} = \left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ $\mathbf{arg}(z) = \mathbf{mes}\left(\widehat{\vec{\imath};\overrightarrow{OM}}\right)$ <u>**Example**</u> $\mathbf{Arg}(2+2i) = \left(\widehat{\frac{\pi}{4}}\right)$ $\mathbf{arg}(2+2i) = \frac{\pi}{4}$ <u>**Remark**</u> If the affix of the vector $\vec{u}$ is $Z_{\vec{u}}$,then $\mathbf{arg}(z_{\vec{u}}) = \mathbf{mes}(\widehat{\vec{\imath};\vec{u}})$ <u>**Polar form of a non-zero complex number**</u> Let $z = a + ib$ be a non-zero complex number	

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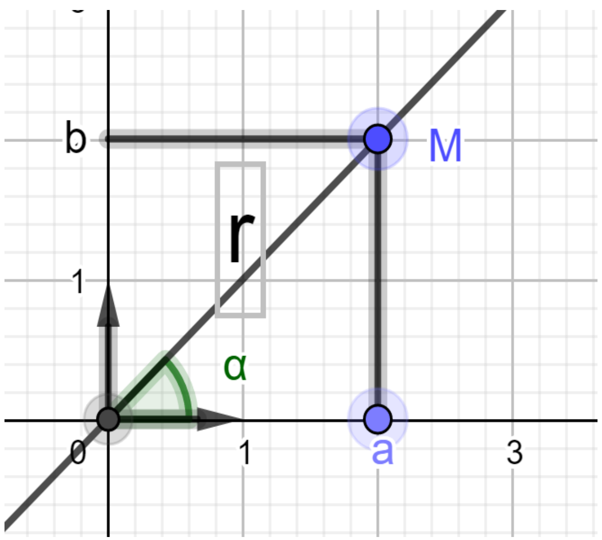
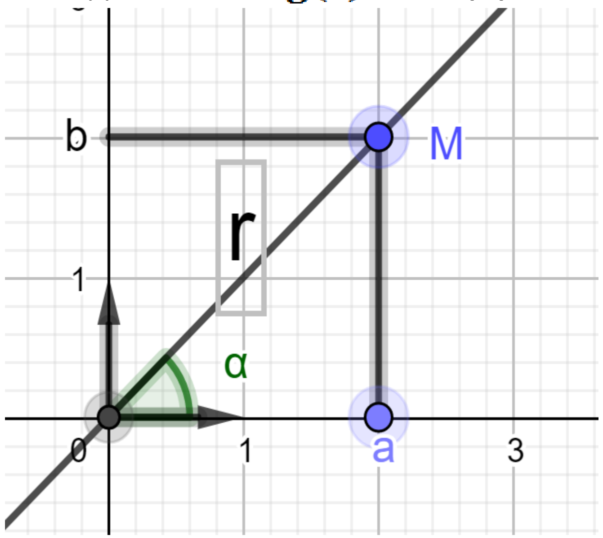
	And M the image of z in an Argand Diagram. We suppose that $\arg(z) = \alpha$, $\lvert z \rvert$=r  Cosα=$\frac{a}{r}$ →a=r Cosα sinα=$\frac{b}{r}$ →b=r sinα $z = r$ Cos$\alpha + ir$ sinα=r(Cos$\alpha + i$sinα) **z=r(Cos$\alpha + i$sinα)** is called the polar			And M the image of z in an Argand Diagram. We suppose that $\arg(z) = \alpha$, $\lvert z \rvert$=r  Cosα=$\frac{a}{r}$ →a=r Cosα sinα=$\frac{b}{r}$ →b=r sinα $z = r$ Cos$\alpha + ir$ sinα=r(Cos$\alpha + i$sinα) **z=r(Cos$\alpha + i$sinα)** is called the polar (trigonometric) form of the complex number z.	

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	(trigonometric) form of the complex number z. This polar form can also be written as follow $\boldsymbol{Z}$=$[\boldsymbol{r};\boldsymbol{\alpha}]$ **<u>Exponential form of a non-zero complex number</u>** let Z=$[r;\alpha]$ be a complex number, the exponential form of z is $\boldsymbol{z=re^{i\alpha}}$ **<u>Remark</u>** If $z=a+ib$, then $\lvert z\rvert$=$\sqrt{a^2+b^2}$ $z=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}+i\frac{b}{\sqrt{a^2+b^2}}\right)$ By definition $Cos\alpha=\frac{a}{\sqrt{a^2+b^2}}$ and $sin\ \alpha=\frac{b}{\sqrt{a^2+b^2}}$ **<u>example :</u>** give the polar and exponential form of the following complex numbers: $z_1=1+i\sqrt{3}$ and $z_2=1+i$ **<u>Resolution</u>** $\lvert z_1\rvert=2$ $z_1=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$ $Cos\alpha=\frac{1}{2}$ and $sin\ \alpha=\frac{\sqrt{3}}{2}$ implies that $\alpha=\frac{\pi}{3}$ Then the **polar form** of			This polar form can also be written as follow $\boldsymbol{Z}$=$[\boldsymbol{r};\boldsymbol{\alpha}]$ **<u>Exponential form of a non-zero complex number</u>** let Z=$[r;\alpha]$ be a complex number, the exponential form of z is $\boldsymbol{z=re^{i\alpha}}$ **<u>Remark</u>** If $z=a+ib$, then $\lvert z\rvert$=$\sqrt{a^2+b^2}$ $z=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}+i\frac{b}{\sqrt{a^2+b^2}}\right)$ By definition $Cos\alpha=\frac{a}{\sqrt{a^2+b^2}}$ and $sin\ \alpha=\frac{b}{\sqrt{a^2+b^2}}$ **<u>example :</u>** give the polar and exponential form of the following complex numbers: $z_1=1+i\sqrt{3}$ and $z_2=1+i$ **<u>Resolution</u>** $\lvert z_1\rvert=2$ $z_1=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$ $Cos\alpha=\frac{1}{2}$ and $sin\ \alpha=\frac{\sqrt{3}}{2}$ implies that $\alpha=\frac{\pi}{3}$ Then the **polar form** of	

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its **exponential form** is Z_1=2$e^{i\frac{\pi}{3}}$
$|z_2| = \sqrt{2}; z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$
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$z_2 = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = \left[\sqrt{2};\frac{\pi}{4}\right]$
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Properties
Let Z=$[r;\alpha] = re^{i\alpha}$ and z'=$[r';\alpha'] = r'e^{i\alpha'}$
be two complex numbers;
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From the properties we can deduce that :</td><td></td><td></td><td>$z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = \left[2;\frac{\pi}{3}\right]$
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$|z_2| = \sqrt{2}; z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$
$Cos\alpha = \frac{\sqrt{2}}{2}$ and $sin\ \alpha = \frac{\sqrt{2}}{2}$ implies that $\alpha = \frac{\pi}{4}$

Then the **polar form** of
$z_2 = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = \left[\sqrt{2};\frac{\pi}{4}\right]$
its **exponential form** is Z_2=$\sqrt{2}e^{i\frac{\pi}{4}}$
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Let Z=$[r;\alpha] = re^{i\alpha}$ and z'=$[r';\alpha'] = r'e^{i\alpha'}$
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	$arg(zz') = arg(z) + arg(z')$ $arg(\frac{z}{z'}) = \arg(z) - \arg(z')$			$arg(zz') = arg(z) + arg(z')$ $arg(\frac{z}{z'}) = \arg(z) - \arg(z')$	
Exercises of Application **(10mins)**	**<u>Exercise</u>** 1)Determine the loci of points $M(z)$ such that $\|z-1+i\| = \|1+3i\|$ $\|z+2-3i\| = \|z+1+i\|$ 2)Given that $z_1 = \frac{\sqrt{6}-i\sqrt{2}}{2}$ and $z_2 = 1-i$ a)Determine the modulus and argument of z_1 and z_2 b) Give the algebraic ,polar and exponential form of the quotient $\frac{z_1}{z_2}$ c) Deduce the values of $cos\frac{\pi}{12}$ and $sin\frac{\pi}{12}$.	Dictates the exercise	Take down the exercise and do it	**<u>Exercise</u>** 1)Determine the loci of points $M(z)$ such that $\|z-1+i\| = \|1+3i\|$ $\|z+2-3i\| = \|z+1+i\|$ 2)Given that $z_1 = \frac{\sqrt{6}-i\sqrt{2}}{2}$ and $z_2 = 1-i$ a)Determine the modulus and argument of z_1 and z_2 b) Give the algebraic ,polar and exponential form of the quotient $\frac{z_1}{z_2}$ c) Deduce the values of $cos\frac{\pi}{12}$ and $sin\frac{\pi}{12}$.	
Conclusion **(5mins)**	**<u>Bilingual game</u>** Give the equivalence of the following words in French: modulus; argument; polar form. **<u>Home work</u>** **<u>Announcement of the next lesson</u>**. The next lesson will be on the square root of a complex number and solutions of quadratic equations.	Copies questions on the board	Copy questions in their note books	**<u>Bilingual game</u>** Give the equivalence of the following words in French: modulus; argument; polar form. **<u>Home work</u>** **<u>Announcement of the next lesson</u>**. The next lesson will be on the square root of a complex number and solutions of quadratic equations	