SAMPLE LESSON: MATHEMATICS

Class: Upper Sixth/ Further Mathematics

Module: Geometry II	**TOPIC:** Complex Numbers
Title of Lesson: Polar form (trigonometric form), exponential form of a non-zero complex number and Euler's formula	**Duration of Lesson:** 120mins

Name of Authors: Inspectorate of Pedagogy/Sciences for the Far North Region

Module 4 : **GEOMETRY II**

Topic: Complex Numbers

Lesson: Polar form (trigonometric form), exponential form of a non-zero complex number and **Euler's formula**

Objectives: At the end of this lesson, the learners should be able to:

1) Calculate the modulus of a complex number
2) Find Argument of a non-zero complex number;
3) Give the polar form of a non-zero complex number;
4) Give the exponential form of a non-zero complex number.
5) **Use Euler's formula in some operations**.

Key question:

What is the polar form of a non-zero complex number?

Prerequisite knowledge:

- ✓ Verify whether students can calculate the norm of a vector.
- ✓ Verify whether students can determine the measurement of the angle between the positive X-axis and a vector.

Motivation: The study of complex numbers comes to reinforce our knowledge and skills necessary to study plane geometry.

Didactic materials

Chalk, colour chalk, Chalkboard, ruler and set square.

REFERENCES

- EWANE ROLAND ALUNGE. Further Pure Mathematics Made Easy Third Edition.

-Further Mathematics Teaching Syllabuses (January 2020)

STAGES/ DURATION	TEACHING/LEARNING ACTIVITIES	TEACHER'S ACTIVITIES	LEARNERS' ACTIVITIES	LEARNING POINTS
INTRODUCTION **5mins**	**<u>Verification of Pre-requisites</u>** **<u>Exercise</u>** a) Calculate the norm of $\overrightarrow{OA}\binom{0}{4}$ given in the Cartesian plane. b) Determine the measure of the angle between $\vec{\imath}$ and $\overrightarrow{OA}$.	-Copies questions on the board -Calls students to the board	-Solve on the board as called by the teacher	**<u>Verification of Pre-requisites</u>** **<u>Exercise</u>** a) Calculate the norm of $\overrightarrow{OA}\binom{0}{4}$ given in the Cartesian plane. b) Determine the measure of the angle between $\vec{\imath}$ and $\overrightarrow{OA}$
Lesson Development and Summary **(100mins)**	**<u>Activity</u>** In an orthonormal reference system (O, $\vec{\imath}$, $\vec{\jmath}$),we consider the points A,B and C with respective affixes $z_A = 2, z_B = 3i$ and $z_C = 2 + 2i$ 1) Plot the points A, B and C in this reference system. 2) Calculate the values of OA,OB and OC. 3) Determine the measure of the following angles: a) The angle between $\vec{\imath}$ and $\overrightarrow{OA}$. b) The angle between $\vec{\imath}$ and $\overrightarrow{OB}$. c) The angle between $\vec{\imath}$ and $\overrightarrow{OC}$.	-Copies activity on the chalkboard -Instructs students to copy in their notebooks and allows them 10 minutes to research	-Follow the instructions and carry out the activity while interacting with each other -Copy notes	**<u>Activity</u>** In an orthonormal reference system (O, $\vec{\imath}$, $\vec{\jmath}$),we consider the points A,B and C with respective affixes $z_A = 2, z_B = 3i$ and $z_C = 2 + 2i$ 1) Plot the points A, B and C in this reference system. 2) Calculate the values of OA,OB and OC. 3) Determine the measure of the following angles: a) The angle between $\vec{\imath}$ and $\overrightarrow{OA}$. b) The angle between $\vec{\imath}$ and $\overrightarrow{OB}$. c) The angle between $\vec{\imath}$ and $\overrightarrow{OC}$.

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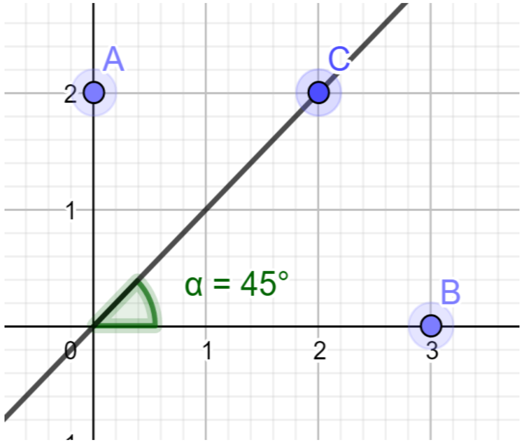
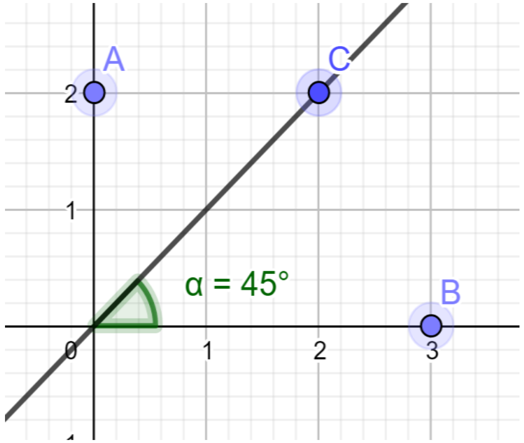
	Resolution 1)  2)OA=$\sqrt{0^2+2^2}$=2,OB$\sqrt{3^2+0^2}$=3 OC=$\sqrt{2^2+2^2}$=$\sqrt{8}$ 3)According to the figure above, a)The angle between $\vec{i}$ and $\overrightarrow{OA}$ is $\frac{\pi}{2}$ b)The angle between $\vec{i}$ and $\overrightarrow{OB}$ is 0 c) The angle between $\vec{i}$ and $\overrightarrow{OC}$ is $\frac{\pi}{4}$. **Observation** For example →)$\sqrt{8}$ Is called modulus of the complex number	-Copies notes on the board -Explains concepts -Dictates notes	in their books Copy notes in their notebooks Follow up as the teacher	**Resolution** 1)  2)OA=$\sqrt{0^2+2^2}$=2,OB$\sqrt{3^2+0^2}$=3 OC=$\sqrt{2^2+2^2}$=$\sqrt{8}$ 3)According to the figure above, a)The angle between $\vec{i}$ and $\overrightarrow{OA}$ is $\frac{\pi}{2}$ b)The angle between $\vec{i}$ and $\overrightarrow{OB}$ is 0 c) The angle between $\vec{i}$ and $\overrightarrow{OC}$ is $\frac{\pi}{4}$. **Observation** For example →)$\sqrt{8}$ Is called modulus of the complex number

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→) $(\vec{i}\,;\overrightarrow{OC}) = \frac{\pi}{4}$ will be called argument of the complex number $z = 2 + 2i$.
<u>Definition</u> (modulus)
Let $z = a + bi$ be a complex number,
We call the modulus of z the strictly positive real number denoted by $|z|$ and define by
$|z|=\sqrt{a^2 + b^2}$
<u>Example</u>
Calculate the modulus of the complex number
$z = 2 + 3i$
<u>Resolution</u> $|z|=\sqrt{2^2 + 3^2} = \sqrt{13}$
<u>Properties</u>
Let z and z' two complex numbers
→) $|z|=0$ if and only if $z = 0$.
→) $|z \times z'|=|z| \times |z'|$
→) $|z^n| = |z|^n \ \forall n \in \mathbb{N}$
→) $|\bar{z}|=|z|$
→) $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|} \ \forall z' \in \mathbb{C}^*$
→) $|\boldsymbol{z} + \boldsymbol{z}'| \leq |\boldsymbol{z}| + |\boldsymbol{z}'|$
→) $|\boldsymbol{z} + \boldsymbol{z}'| \geq ||\boldsymbol{z}| - |\boldsymbol{z}'||$

<u>Remark</u>
For any $M(z)$ and $M'(z')$
$\boldsymbol{MM' = |Z_{M'} - Z_M|}$
<u>Definition</u> (argument)</td><td>Solves the examples one after the other while questioning the students</td><td>explains, asking and answering questions where necessary</td><td>$z = 2 + 2i$.
→) $(\vec{i}\,;\overrightarrow{OC}) = \frac{\pi}{4}$ will be called argument of the complex number $z = 2 + 2i$.
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→) $|z + z'| \leq |z| + |z'|$
→) $|z + z'| \geq ||z| - |z'||$

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For any $M(z)$ and $M'(z')$
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STAGES/ DURATION	TEACHING/LEARNING ACTIVITIES	TEACHER'S ACTIVITIES	LEARNERS' ACTIVITIES	LEARNING POINTS
	Let z be a non-zero complex number, M the image of the complex number z ,in the complex plane with an orthonormal reference system $(O, \vec{\imath}, \vec{\jmath})$, -We call **Argument** of z the **angle** $(\widehat{\vec{\imath}; \overrightarrow{OM}})$ denoted $\boldsymbol{Arg(z)}$ - We call **argument** of z the **measure** of **angle** $(\widehat{\vec{\imath}; \overrightarrow{OM}})$ denoted $\boldsymbol{arg(z)}$ $\mathbf{Arg(z)} = (\widehat{\vec{\imath}; \overrightarrow{OM}})$ $\mathbf{arg}(z) = \mathbf{mes}(\widehat{\vec{\imath}; \overrightarrow{OM}})$ **Example** $\mathbf{Arg}(2+2i) = (\widehat{\frac{\pi}{4}})$ $\mathbf{arg}(2+2i) = \frac{\pi}{4}$ **Remark** If the affix of the vector $\vec{u}$ is $Z_{\vec{u}}$,then $\mathbf{arg}(z_{\vec{u}}) = \mathbf{mes}(\widehat{\vec{\imath}; \vec{u}})$ **Polar form of a non-zero complex number** Let $z = a + ib$ be a non-zero complex number And M the image of z in an Argand Diagram. We suppose that $\mathbf{arg}(Z) = \alpha$, **|z|=r**			Let z be a non-zero complex number, M the image of the complex number z ,in the complex plane with an orthonormal reference system $(O, \vec{\imath}, \vec{\jmath})$, -We call **Argument** of z the **angle** $(\widehat{\vec{\imath}; \overrightarrow{OM}})$ denoted $\boldsymbol{Arg(z)}$ - We call **argument** of z the **measure** of **angle** $(\widehat{\vec{\imath}; \overrightarrow{OM}})$ denoted $\boldsymbol{arg(z)}$ $\mathbf{Arg(z)} = (\widehat{\vec{\imath}; \overrightarrow{OM}})$ $\mathbf{arg(z)} = \mathbf{mes}(\widehat{\vec{\imath}; \overrightarrow{OM}})$ **Example** $\mathbf{Arg}(2+2i) = (\widehat{\frac{\pi}{4}})$ $\mathbf{arg}(2+2i) = \frac{\pi}{4}$ **Remark** If the affix of the vector $\vec{u}$ is $Z_{\vec{u}}$,then $\mathbf{arg}(z_{\vec{u}}) = \mathbf{mes}(\widehat{\vec{\imath}; \vec{u}})$ **Polar form of a non-zero complex number**

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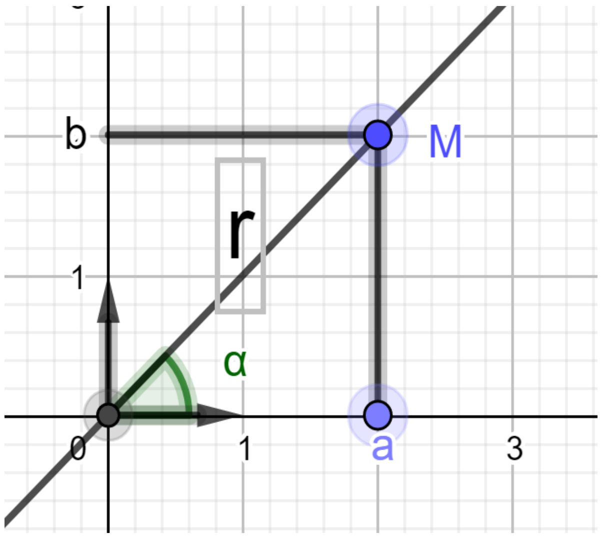
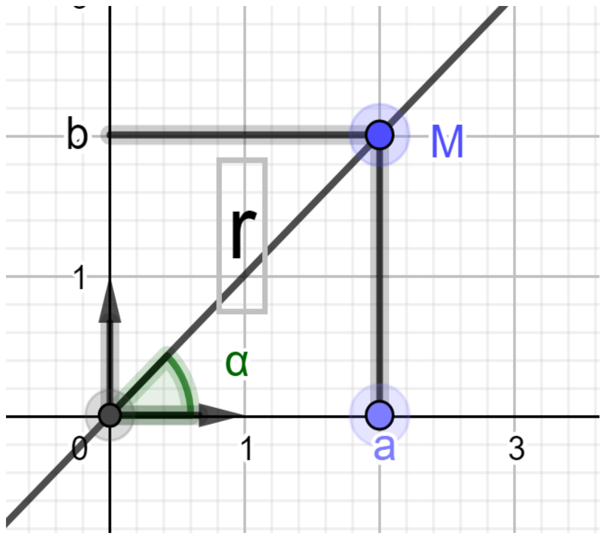
	 Cosα=$\frac{a}{r}$ →a=r Cosα sinα=$\frac{b}{r}$ →b=r sinα $z = r\ \mathrm{Cos}\alpha + ir\ \sin\alpha$=r($\mathrm{Cos}\alpha + i\sin\alpha$) $\boldsymbol{z}$=$\boldsymbol{r}$($\boldsymbol{\mathrm{Cos}\alpha + i\sin\alpha}$) is called the polar (trigonometric) form of the complex number z. This polar form can also be written as follow $\boldsymbol{z}$=$[r;\alpha]$			Let $z = a + ib$ be a non-zero complex number And M the image of z in an Argand Diagram. We suppose that $\boldsymbol{\arg(z) = \alpha}$, **|z|=r**  Cosα=$\frac{a}{r}$ →a=r Cosα sinα=$\frac{b}{r}$ →b=r sinα $z = r\ \mathrm{Cos}\alpha + ir\ \sin\alpha$=r($\mathrm{Cos}\alpha + i\sin\alpha$) $\boldsymbol{z}$=$\boldsymbol{r}$($\boldsymbol{\mathrm{Cos}\alpha + i\sin\alpha}$) is called the polar

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	<u>Exponential form of a non-zero complex number</u> let Z=$[r\,;\alpha]$ be a complex number, the exponential form of z is $\boldsymbol{z = re^{i\alpha}}$ **<u>Remark</u>** If $z = a + ib$, then $\lvert z\rvert$=$\sqrt{a^2+b^2}$ $z = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right)$ By definition $Cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $sin\ \alpha = \frac{b}{\sqrt{a^2+b^2}}$ **<u>example :</u>** give the polar and exponential form of the following complex numbers: $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 + i$ **<u>Resolution</u>** $\lvert z_1\rvert = 2$ $z_1 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $Cos\alpha = \frac{1}{2}$ and $sin\ \alpha = \frac{\sqrt{3}}{2}$ implies that $\alpha = \frac{\pi}{3}$ Then the **polar form** of $z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = \left[2;\frac{\pi}{3}\right]$ its **exponential form** is Z_1=$2e^{i\frac{\pi}{3}}$ $\lvert z_2\rvert = \sqrt{2}; z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$			(trigonometric) form of the complex number z. This polar form can also be written as follow $\boldsymbol{Z}$=$[\boldsymbol{r};\boldsymbol{\alpha}]$ **<u>Exponential form of a non-zero complex number</u>** let Z=$[r\,;\alpha]$ be a complex number, the exponential form of z is $\boldsymbol{z = re^{i\alpha}}$ **<u>Remark</u>** If $z = a + ib$, then $\lvert z\rvert$=$\sqrt{a^2+b^2}$ $z = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right)$ By definition $Cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $sin\ \alpha = \frac{b}{\sqrt{a^2+b^2}}$ **<u>example :</u>** give the polar and exponential form of the following complex numbers: $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 + i$ **<u>Resolution</u>** $\lvert z_1\rvert = 2$ $z_1 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $Cos\alpha = \frac{1}{2}$ and $sin\ \alpha = \frac{\sqrt{3}}{2}$ implies that $\alpha = \frac{\pi}{3}$ Then the **polar form** of $z_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = \left[2;\frac{\pi}{3}\right]$ its **exponential form** is Z_1=$2e^{i\frac{\pi}{3}}$

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$Cos\alpha = \frac{\sqrt{2}}{2}$ and $sin\ \alpha = \frac{\sqrt{2}}{2}$ implies that $\alpha = \frac{\pi}{4}$

Then the **polar form** of

$z_2 = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = \left[\sqrt{2}; \frac{\pi}{4}\right]$

its **exponential form** is Z_2=$\sqrt{2}e^{i\frac{\pi}{4}}$

<u>Properties</u>

Let Z=$[r; \alpha] = re^{i\alpha}$ and z'=$[r'; \alpha'] = r'e^{i\alpha'}$ be two complex numbers;

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Remark

From the properties we can deduce that :

$arg(zz') = arg(z) + arg(z')$

$arg(\frac{z}{z'}) = \arg(z) - \arg(z')$

<u>Euler's formula</u>

$\boldsymbol{cos\theta} = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and

$\boldsymbol{sin\theta} = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ for any real number θ.

</td><td></td><td></td><td>

$|z_2| = \sqrt{2}; z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$Cos\alpha = \frac{\sqrt{2}}{2}$ and $sin\ \alpha = \frac{\sqrt{2}}{2}$ implies that $\alpha = \frac{\pi}{4}$

Then the **polar form** of

$z_2 = \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right) = \left[\sqrt{2}; \frac{\pi}{4}\right]$

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				$\boldsymbol{sin\theta} = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ for any real number θ.
Exercises of Application **(10mins)**	**<u>Exercise</u>** 1)Determine and **represent t**he loci of points $M(z)$ such that: $\lvert z-1+i\rvert = \lvert 1+3i\rvert$ $\lvert z+2-i\rvert = \lvert z+i\rvert$ $\boldsymbol{\lvert z+2-3i\rvert = 2\lvert z+1+i\rvert}$ **arg**$\boldsymbol{(z-1+i)=\frac{\pi}{4}}$ 2)Given that $z_1=\frac{\sqrt{6}-i\sqrt{2}}{2}$ and $z_2 = 1-i$ a)Determine the modulus and argument of z_1 and z_2 b) Give the algebraic ,polar and exponential form of the quotient $\frac{z_1}{z_2}$ c) Deduce the values of $cos\frac{\pi}{12}$ and $sin\frac{\pi}{12}$. 3)**Use Euler's formula** to express $cos^4 x$ in terms of cos4x and $cosx$ and deduce the value of the integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos^4 xdx$	Dictates the exercise	Take down the exercise and do it	**<u>Exercise</u>** 1)Determine and represent the loci of points $M(z)$ such that: $\lvert z-1+i\rvert = \lvert 1+3i\rvert$ $\lvert z+2-i\rvert = \lvert z+i\rvert$ $\lvert z+2-3i\rvert = 2\lvert z+1+i\rvert$ arg$(z-1+i)=\frac{\pi}{4}$ 2)Given that $z_1=\frac{\sqrt{6}-i\sqrt{2}}{2}$ and $z_2 = 1-i$ a)Determine the modulus and argument of z_1 and z_2 b) Give the algebraic ,polar and exponential form of the quotient $\frac{z_1}{z_2}$ c) Deduce the values of $cos\frac{\pi}{12}$ and $sin\frac{\pi}{12}$. 3)Use Euler's formula to express $cos^4 x$ in terms of cos4x and $cosx$ and deduce the value of the integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos^4 xdx$
Conclusion **(5mins)**	**<u>Bilingual game</u>** Give the equivalence of the following words in French: modulus; argument; polar form; Euler formula. **<u>Home work</u>**	Copies questions on the board	Copy questions in their note books	**<u>Bilingual game</u>** Give the equivalence of the following words in French: modulus; argument; polar form;Euler formula.

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	Announcement of the next lesson. The next lesson will be on the nth-root of a complex number and their representation on the Argand diagram and solutions of quadratic equations.			**Home work** **Announcement of the next lesson**. The next lesson will be on the nth-root of a complex number and their representation on the Argand diagram and solutions of quadratic equations.