# SAMPLE LESSON: MATHEMATICS

## Class: Upper Sixth/ Further Mathematics

| **Module:** Geometry II | **TOPIC:** Complex Numbers |
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| **Title of Lesson:** $n^{th}$ root of a non-zero complex number and their representations on the complex plane, quadratic equation and De Moivre's Theorem. | **Duration of Lesson:** 120mins |

**Name of Authors:** Inspectorate of Pedagogy/Sciences for the Far North Region

**Module** 4 : **GEOMETRY II**

**Topic:** Complex Numbers

**Lesson**: $n^{th}$ root of a non-zero complex number and their representation on the complex plane, quadratic equation and De Moivre's Theorem.

**Objectives:** At the end of this lesson, the learners should be able to:

1) Determine the algebraic form of the square root of a non-zero complex number ;
2) Find the $n^{th}$ root of a non-zero complex number and **represent the images** of affixes.
3) Find solutions of a quadratic equation in $\mathbb{C}$ and also find the roots of polynomial equations with real coefficients;
4) Use De Moivre's Theorem to express
   i) $sinn\theta, cosn\theta$ and $tann\theta$ in terms of powers of $sin\theta, cos\theta$ and $tan\theta$
   ii) Power of sine or cosine of angles in terms of sine and cosine of multiple angles.

**Key question:**

How can we solve a polynomial equation in $\mathbb{C}$ ?

**Prerequisite knowledge:**

- ✓ Verify whether students can calculate the modulus of a complex number.
- ✓ Verify whether students can solve simultaneous linear equations.

**Motivation**: The study of complex numbers comes to reinforce our knowledge and skills necessary to study plane geometry.

**Didactic materials**

Chalk ,colour chalk, Chalkboard,ruler and set square.

**REFERENCES**

- EWANE ROLAND ALUNGE. Further Mathematics Made Easy Third Edition.
-Further Mathematics Teaching Syllabuses(January 2020)

| STAGES/ DURATION | TEACHING/LEARNING activities | TEACHER'S ACTIVITIES | LEARNERS' ACTIVITIES | LEARNING POINTS |
|---|---|---|---|---|
| **Introduction**<br>**5mins** | <u>**Verification of Pre-requisites**</u><br><u>**Exercise**</u><br>a) Calculate the modulus of the complex number $z = 4 - 7i$<br>b) Solve the simultaneous linear equation: $\begin{cases} x + 2y = 2 \\ x - 2y = -4 \end{cases}$ | -Copies questions on the board<br>-Calls students to the board | -Solve on the board , while the others follow up | <u>**Verification of Pre-requisites**</u><br><u>**Exercise**</u><br>a) Calculate the modulus of the complex number $z = 4 - 7i$<br>b) Solve the simultaneous linear equation: $\begin{cases} x + 2y = 2 \\ x - 2y = -4 \end{cases}$ |
| **Lesson Development and Summary**<br>**(100mins)** | <u>**Activity 1**</u><br>Given a complex number $z = 3 - 4i$<br>1)Calculate $(-2 + i)^2$.what can we say about $z$ and $b = -2 + i$<br>2)Let $\delta = x + iy$ such that $\delta^2 = z$.<br>a)What can we say about $\delta$ and $z$<br>b)Give a relationship between $\lvert\delta^2\rvert$ and $\lvert z\rvert$<br>c) Using the previous information ,find the values $(x_1,y_1)$ and $(x_2,y_2)$ of $(x, y)$.<br>d) Consider $\delta_1 = x_1 + iy_1$ and $\delta_2 = x_2 + iy_2$<br>Calculate ${\delta_1}^2$ , ${\delta_2}^2$ and conclude.<br><u>**Resolution**</u><br>1)we easily verify that $(-2 + i)^2 = 3 - 4i$,<br>We can say that $b$ is a square root of $z$.<br>2)a) We can say that $\delta$ is a square root of $z$.<br>b) The relationship between $\lvert\delta^2\rvert$ and $\lvert z\rvert$ is $\lvert\delta^2\rvert$ =$\lvert z\rvert$ because $\delta^2 = z$.<br>c) Let's find the values $(x_1,y_1)$ and $(x_2,y_2)$ of $(x, y)$.<br>We have<br>$\lvert\delta^2\rvert$ =$\lvert z\rvert \leftrightarrow x^2 + y^2$=5<br>$\delta^2 = z \leftrightarrow x^2 - y^2 + i2xy = 3 - 4i$<br>By identification, $x^2 - y^2 = 3$ and $2xy = -4$ | -Copies activity on the chalkboard<br>-Instructs students to copy in their notebooks and allows them 10 minutes to research<br><br>-Copies notes on the board<br><br>-Explains concepts | -Follow the instructions and carry out the activity while interacting with each other<br><br>-Copy notes in their books | <u>**Activity 1**</u><br>Given a complex number $z = 3 - 4i$<br>1)Calculate $(-2 + i)^2$.what can we say about $z$ and $b = -2 + i$<br>2)Let $\delta = x + iy$ such that $\delta^2 = z$.<br>a)What can we say about $\delta$ and $z$<br>b)Give a relationship between $\lvert\delta^2\rvert$ and $\lvert z\rvert$<br>c) Using the previous information ,find the values $(x_1,y_1)$ and $(x_2,y_2)$ of $(x, y)$.<br>d) Consider $\delta_1 = x_1 + iy_1$ and $\delta_2 = x_2 + iy_2$<br>Calculate ${\delta_1}^2$ , ${\delta_2}^2$ and conclude.<br><u>**Resolution**</u><br>1)we easily verify that $(-2 + i)^2 = 3 - 4i$,<br>We can say that $b$ is a square root of $z$.<br>2)a) We can say that $\delta$ is a square root of $z$.<br>b) The relationship between $\lvert\delta^2\rvert$ and $\lvert z\rvert$ is $\lvert\delta^2\rvert$ =$\lvert z\rvert$ because $\delta^2 = z$.<br>c) Let's find the values $(x_1,y_1)$ and $(x_2,y_2)$ of $(x, y)$.<br>We have<br>$\lvert\delta^2\rvert$ =$\lvert z\rvert \leftrightarrow x^2 + y^2$=5<br>$\delta^2 = z \leftrightarrow x^2 - y^2 + i2xy = 3 - 4i$<br>By identification, $x^2 - y^2 = 3$ and $2xy = -4$ |

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| | Then, $\begin{cases} 2xy = -4 \textbf{ is called sign equation} \\ x^2+y^2=5 \\ x^2-y^2=3 \end{cases}$<br>$\begin{cases} x^2+y^2=5 \\ x^2-y^2=3 \end{cases}$ then $y^2=1$ and $y=1$ or $y=-1$<br>$x^2=4$ and $x=2$ or $x=-2$<br><br>$2xy=-4$ implies that $x$ and $y$ should be opposite in sign<br>If $y=1$ then $x=-2$, $(x_1,y_1)=(-2;1)$<br>If $y=-1$ then $x=2$, $(x_2,y_2)=(2;-1)$<br>d) $\delta_1=-2+i$ and $\delta_2=2-i$<br>${\delta_1}^2=(-2+i)^2=3-4i$,<br>${\delta_2}^2=(2-i)^2=3-4i$<br>**Conclusion**<br>$\delta_1$ and $\delta_2$ are the square roots of $z=3-4i$<br><br>**Activity 2**<br>Given the complex number $z$=1+i$\sqrt{3}$ .Give the polar form of $z$ and deduce the Cartesian form of $z^3$.<br>**Resolution**<br>$z=1+i\sqrt{3}=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})=\left[2;\frac{\pi}{3}\right]$<br>$z^3=$<br>$\left[2;\frac{\pi}{3}\right]^3=\left[2^3;3\times\frac{\pi}{3}\right]=[8;\pi]=8(\cos\pi+i\sin\pi)=-8$<br><br>$z^3=-8$ we say that $1+i\sqrt{3}$ is a cube root of $-8$. | -Dictates notes<br><br>Solves the examples one after the other while questioning the students | Copy notes in their notebooks<br><br>Follow up as the teacher explains, asking and answering questions where necessary | Then, $\begin{cases} 2xy = -4 \textbf{ is called sign equation} \\ x^2+y^2=5 \\ x^2-y^2=3 \end{cases}$<br>$\begin{cases} x^2+y^2=5 \\ x^2-y^2=3 \end{cases}$ then $y^2=1$ and $y=1$ or $y=-1$<br>$x^2=4$ and $x=2$ or $x=-2$<br><br>$2xy=-4$ implies that $x$ and $y$ should be opposite in sign<br>If $y=1$ then $x=-2$, $(x_1,y_1)=(-2;1)$<br>If $y=-1$ then $x=2$, $(x_2,y_2)=(2;-1)$<br>d) $\delta_1=-2+i$ and $\delta_2=2-i$<br>${\delta_1}^2=(-2+i)^2=3-4i$,<br>${\delta_2}^2=(2-i)^2=3-4i$<br>**Conclusion**<br>$\delta_1$ and $\delta_2$ are the square roots of $z=3-4i$<br><br>**Activity 2**<br>Given the complex number $z$=1+i$\sqrt{3}$ .Give the polar form of $z$ and deduce the Cartesian form of $z^3$.<br>**Resolution**<br>$z=1+i\sqrt{3}=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})=\left[2;\frac{\pi}{3}\right]$<br>$z^3=$<br>$\left[2;\frac{\pi}{3}\right]^3=\left[2^3;3\times\frac{\pi}{3}\right]=[8;\pi]=8(\cos\pi+i\sin\pi)=-8$<br><br>$z^3=-8$ we say that $1+i\sqrt{3}$ is a cube root of $-8$. |

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| | <u>**Definition**</u><br>-The $n^{th}$ root of a complex number $Z$ is any complex number $z$ such that<br>$z^n = Z$<br>- The square root of a complex number $Z$ is any complex number $z$ such that<br>$z^2 = Z$<br><u>**Determination of the $n^{th}$ root in polar form of a complex number.**</u><br>Let $Z=[r;\theta]$ ;$z=[\rho;\alpha]$ be two complex numbers.<br>$z^n = Z \quad \leftrightarrow [\rho;\alpha]^n = [r;\theta]$<br>$\leftrightarrow [\rho^n; n\alpha] = [r;\theta]$<br>$\leftrightarrow \begin{cases} \rho^n = r \\ n\alpha = \theta + k2\pi \end{cases}$<br>$\leftrightarrow \begin{cases} \rho = \sqrt[n]{r} \\ \alpha = \frac{\theta}{n} + \frac{2k\pi}{n} \end{cases} \quad \boldsymbol{k \in \{0;1..n-1\}}$<br>$z_k=\left[\sqrt[n]{r};\frac{\theta}{n}+\frac{2k\pi}{n}\right], \boldsymbol{k \in \{0;1..n-1\}}$ are the polar forms of $n^{th}$ root of the complex number $\boldsymbol{Z}$.<br>**NB:**The images of $z_k=\left[\sqrt[n]{r};\frac{\theta}{n}+\frac{2k\pi}{n}\right], k \in \{0;1..n-1\}$ | | | <u>**Definition**</u><br>-The $n^{th}$ root of a complex number $Z$ is any complex number $z$ such that<br>$z^n = Z$<br>- The square root of a complex number $Z$ is any complex number $z$ such that<br>$z^2 = Z$<br><u>**Determination of the $n^{th}$ root in polar form of a complex number.**</u><br>Let $Z=[r;\theta]$ ;$z=[\rho;\alpha]$ be two complex numbers.<br>$z^n = Z \quad \leftrightarrow [\rho;\alpha]^n = [r;\theta]$<br>$\leftrightarrow [\rho^n; n\alpha] = [r;\theta]$<br>$\leftrightarrow \begin{cases} \rho^n = r \\ n\alpha = \theta + k2\pi \end{cases}$<br>$\leftrightarrow \begin{cases} \rho = \sqrt[n]{r} \\ \alpha = \frac{\theta}{n} + \frac{2k\pi}{n} \end{cases} \quad \boldsymbol{k \in \{0;1..n-1\}}$<br>$z_k=\left[\sqrt[n]{r};\frac{\theta}{n}+\frac{2k\pi}{n}\right], \boldsymbol{k \in \{0;1..n-1\}}$ are the polar forms of $n^{th}$ root of the complex number $\boldsymbol{Z}$.<br>**NB:**The images of $z_k=\left[\sqrt[n]{r};\frac{\theta}{n}+\frac{2k\pi}{n}\right], k \in \{0;1..n-1\}$ |

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| | Are situated on the circle of center O with radius $\sqrt[n]{r}$.<br>These images are the vertices of an n-sided regular polygon.<br>**Definition**<br>A quadratic equation in $\mathbb{C}$,is an equation of the form $az^2 + bz + c = 0$ where a,b and c are elements of $\mathbb{C}$ and a$\neq 0$<br>**Example**<br>(2+i) $z^2 + iz + 1 + i = 0$<br>**Procedure of resolution of the quadratic equation**.<br>→)Calculate the discriminant $\Delta = b^2 + 4ac$;<br>→)Find the algebraic form of the square roots $\delta_1$ and $\delta_2$ of $\Delta$; →)The solution of our equation will be $z_1 = \frac{-b+\delta_1}{2a}$ and $z_2 = \frac{-b+\delta_2}{2a}$<br>$$S = \left\{\frac{-b+\delta_1}{2a} ; \frac{-b+\delta_2}{2a}\right\}$$<br>**Remark** $\delta_1 = -\delta_2$<br>**Definition**<br>A polynomial equation of degree 3 in $\mathbb{C}$,is an equation of the form $az^3 + bz^2 + cz + d = 0$ where a,b , c and d are elements of $\mathbb{C}$ and a$\neq 0$<br>**Example**<br>-5 $z^3 + 4z^2 - \frac{2}{3}z - 3 = 0$<br>**De Moivre's Theorem**<br>$\forall n \in \mathbb{Z}, (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ | | | Are situated on the circle of center O with radius $\sqrt[n]{r}$.<br>These images are the vertices of an n-sided regular polygon.<br>**Definition**<br>A quadratic equation in $\mathbb{C}$,is an equation of the form $az^2 + bz + c = 0$ where a,b and c are elements of $\mathbb{C}$ and a$\neq 0$<br>**Example**<br>(2+i) $z^2 + iz + 1 + i = 0$<br>**Procedure of resolution of the quadratic equation**.<br>→)Calculate the discriminant $\Delta = b^2 + 4ac$;<br>→)Find the algebraic form of the square roots $\delta_1$ and $\delta_2$ of $\Delta$; →)The solution of our equation will be $z_1 = \frac{-b+\delta_1}{2a}$ and $z_2 = \frac{-b+\delta_2}{2a}$<br>$$S = \left\{\frac{-b+\delta_1}{2a} ; \frac{-b+\delta_2}{2a}\right\}$$<br>**Remark** $\delta_1 = -\delta_2$<br>**Definition**<br>A polynomial equation of degree 3 in $\mathbb{C}$,is an equation of the form $az^3 + bz^2 + cz + d = 0$ where a,b , c and d are elements of $\mathbb{C}$ and a$\neq 0$<br>**Example**<br>-5 $z^3 + 4z^2 - \frac{2}{3}z - 3 = 0$<br>**De Moivre's Theorem**<br>$\forall n \in \mathbb{Z}, (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ |
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| **Application Exercises**<br><br>**10mins** | **Exercise**<br>1) Determine the fifth roots of the complex number $z$ in the following cases and **represent their images** on the complex plane.<br>a) $z$ is unity<br>b) $z=-1+i\sqrt{3}$<br>2)Given the following quadratic equation:<br>$(E): z^2+(2+3i)z-2(1-2i)=0$<br>a) Calculate the discriminant $\Delta$ of the equation.<br>b) Find the algebraic form of the square roots of $\Delta$.<br>c) Deduce the solutions of (E).<br>3) consider the polynomial $p$ define by<br>$p(z) = z^3-2z^2-3z+10$ for any complex number $z$.<br>a)verify that -2 is a root of $p$.<br>b)Find the values of a,b and c such that<br>$p(z) = (z+2)(az^2+bz+c)$<br>c)Solve the equation $p(z) = 0$ in $\mathbb{C}$.<br>4)a)Using De Moivre's Theorem,show that:<br>$\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{1999}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$<br>b)Expand the expression,$(cosx+isinx)^3$ and express $cos\,3x$ and $sin3x$ in terms of $cosx$ and $sinx$. | Dictates the exercise | Take down the exercise and work individually | **Exercise**<br>1) Determine the fifth roots of the complex number $z$ in the following cases and **represent their images** on the complex plane.<br>a) $z$ is unity<br>b) $z=-1+i\sqrt{3}$<br>2)Given the following quadratic equation:<br>$(E): z^2+(2+3i)z-2(1-2i)=0$<br>a) Calculate the discriminant $\Delta$ of the equation.<br>b) Find the algebraic form of the square roots of $\Delta$.<br>c) Deduce the solutions of (E).<br>3) consider the polynomial $p$ define by<br>$p(z) = z^3-2z^2-3z+10$ for any complex number $z$.<br>a)verify that -2 is a root of $p$.<br>b)Find the values of a,b and c such that<br>$p(z) = (z+2)(az^2+bz+c)$<br>c)Solve the equation $p(z) = 0$ in $\mathbb{C}$.<br>4)a)Using De Moivre's Theorem,show that:<br>$\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{1999}=\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$<br>b)Expand the expression,$(cosx+isinx)^3$ and express $cos\,3x$ and $sin3x$ in terms of $cosx$ and $sinx$. |
| **CONCLUSION**<br><br>**5 mins** | **Bilingual game**<br>Give the equivalence of the following words in French: unity; square roots; De Moivre's Theorem.<br>**Home work**<br><br>**Announcement of the next lesson**.<br>The next lesson will be on complex number and | Copies questions on the board | Copy questions in their note books | **Bilingual game**<br>Give the equivalence of the following words in French: unity; square roots; De Moivre's Theorem.<br>**Home work**<br><br>**Announcement of the next lesson**.<br>The next lesson will be on complex number and |

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