SAMPLE LESSON: MATHEMATICS

Class: Lower sixth

Title of Module: Plane Geometry

Title of Chapter: Derivatives

Title of Lesson: Differentiation of implicit functions

Duration of Lesson: **90mins**

Name of Authors: Inspectorate of Pedagogy for the North West Region

Class: Lower sixth

Subject: Mathematics

Module 3: Plane Geometry

Topic: Derivatives

Lesson: Differentiation of implicit functions **Duration:** 90 minutes

Lesson Objectives: By the end of the lesson learners should be able to:

- Find the derivatives of functions (in two dimensions) expressed implicitly.
- Use different rules of differentiation to differentiate a combination of implicit functions.

Prerequisite knowledge:

- Knowledge on differentiation of algebraic functions with good understanding of chain rule, product rule and quotient rule.
- Higher derivatives of algebraic functions.

References: - Pure and Applied Maths II by A Dawson and R Parson, 1988

- Pure Maths by Anucam, 2010, Core Course.
- A/L Maths made easy by Ewane, 2017
- National Syllabus, scheme of work
- Core Course, Bostock and Chandler
- An Intergrated Core Approach, Piankeh Albert

Stage/ Duration	Teaching/Learning Activities	Learning points
Introduction **(15 Minutes)**	**Verification of prerequisite knowledge** 1. Differentiate each of the following with respect to x: (a) $3x^2 - x + 5$ (b) $\frac{x^2+2}{2x-3}$ (c) $2x^2\sqrt{x-3}$ 2. Find $\frac{d^2y}{dx^2}$ given that $y = 2 + 3x^2 - x^3$ **Problem situation** After the lessons on differentiation of algebraic functions, the chain rule, the product rule Afeminui was so frustrated because she could not do the following: Given that $x + 3xy - y^3 = 5$, show that $(x - y^2)\frac{d^2y}{dx^2} - 2y\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) = 0$ How can you help Afeminui out of her frustration?	1. Differentiating with respect to x: (a) $\frac{d}{dx}(3x^2 - x + 5) = 6x - 1$ (b) $\frac{d}{dx}\left(\frac{x^2+2}{2x-3}\right) = \frac{(2x-3)(2x)-(x^2+2)(2)}{(2x-3)^2}$ $= \frac{2x^2-6x-4}{(2x-3)^2}$ (c) $\frac{d}{dx}\left(2x^2\sqrt{x-3}\right)$ $= 2x^2\left(\frac{1}{2}\right)(x-3)^{-\frac{1}{2}} + 4x\sqrt{x-3}$ $= \frac{x^2}{\sqrt{x-3}} + 4x\sqrt{x-3}$ 2. $y = 2 + 3x^2 - x^3 \Rightarrow \frac{dy}{dx} = 6x - 3x^2$ $\Rightarrow \frac{d^2y}{dx^2} = 6 - 6x$ A different technique is required to differentiate functions of this form.
Lesson Development **(25 minutes)**	**Definition:** Explicit and implicit functions **Activity: Differentiating $x = y^2$** **Instructions:** 1. Differentiate with respect to y and use the relation $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$ to find $\frac{dy}{dx}$ in terms of y. 2. Rearrange $x = y^2$ to give $y = x^{\frac{1}{2}}$ and differentiate	**Definition:** A function can be explicit or implicit. - An explicit function is one in which one variable can be expressed solely in terms of the other variable e.g (i) $y = x^2 + 2x$ (ii) $y = \sqrt{x+1}$ etc - An implicit function is one in which one variable is not expressed solely in terms of another variable e.g (i) $y = xy + 2y$ (ii) $x^2 + y^2 + 2xy = 0$ etc. 1. $x = y^2 \Rightarrow \frac{dx}{dy} = 2y$ and $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = \frac{1}{2y}$ 2. Rearranging, $x = y^2 \Rightarrow y = x^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2y}$ as

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	with respect to x expressing $\frac{dy}{dx}$ in terms of y. 3. Differentiate both sides of $x = y^2$ with respect to x to have $1 = \frac{d}{dx}(y^2)$................(1) Let $u = y^2$ (a) Find $\frac{du}{dy}$ (b) Using $\frac{du}{dx}=\frac{du}{dy}.\frac{dy}{dx}$ find an expression for $\frac{du}{dx}$ (c) Using the result in (b) find $\frac{d}{dx}(y^2)$ in terms of y and $\frac{dy}{dx}$. (d) Substitute in (1) and and rearrange to obtain $\frac{dy}{dx}$ in terms of y 4. What do you observe?	before. 3. $x = y^2 \Rightarrow \frac{dx}{dx} = \frac{d}{dx}(y^2) \Rightarrow 1 = \frac{d}{dx}(y^2)$..........(1) Let $u = y^2$ (a) $u = y^2 \Rightarrow \frac{du}{dy} = 2y$ (b) $\frac{du}{dx} = \left(\frac{du}{dy}\right)\left(\frac{dy}{dx}\right) = 2y\frac{dy}{dx}$ (c) Hence $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ (d) Substituting in (1) gives $1 = \frac{d}{dx}(y^2) \Rightarrow 1 = 2y\frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = \frac{1}{2y}$ as before 4. **Observations:** - The results in the three steps are the same - $x = y^2 \Rightarrow 1 = \frac{d}{dx}(y^2) \Rightarrow 1 = 2y\frac{dy}{dx}$. This is implicit differentiation. **Rule:** $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$. To differentiate y^2, we differentiate it with respect to y and multiply by $\frac{dy}{dx}$.
Application Exercise (30 minutes)	**Exercise** 1. Differentiate $2xy - x^2 + y^2 = 5$ with respect to x. 2. Find the gradient of the curve $y - \frac{y}{x} + 3x = 5$, $x \in \mathbb{R} - \{0,\ 1\}$ at the point $(2,\ -2)$. 3. Given that $x^2 + 3xy + y^2 = 7$ show that $(3x + 2y)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 6\left(\frac{dy}{dx}\right) + 2 = 0$	**Exercise** 1. Differentiate $2xy - x^2 + y^2 = 5$ with respect to x. 2. Find the gradient of the curve $y - \frac{y}{x} + 3x = 5$, $x \in \mathbb{R} - \{0,\ 1\}$ at the point $(2,\ -2)$. 3. Given that $x^2 + 3xy + y^2 = 7$ show that $(3x + 2y)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 6\left(\frac{dy}{dx}\right) + 2 = 0$ **Solution** 1. $2xy - x^2 - y^2 = 5$

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		$\Rightarrow \frac{d}{dx}(2xy) - \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$ $\Rightarrow 2x\frac{dy}{dx} + y(2) - 2x + 2y\frac{dy}{dx} = 0$ $\Rightarrow 2(x+y)\frac{dy}{dx} = 2(x-y)$ $\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$ Note that $\frac{d}{dx}(2xy)$ is differentiated as a product. 2. $y - \frac{y}{x} + 3x = 8$ $\Rightarrow \frac{dy}{dx} - \left[y(-x^{-2}) + \left(\frac{1}{x}\right)\frac{dy}{dx}\right] + \frac{dy}{dx} + 3 = 0$ $\Rightarrow \left(\frac{1}{x} - 1\right)\frac{dy}{dx} = 3 + \frac{y}{x^2}$ $\Rightarrow \frac{dy}{dx} = \left(\frac{3x^2+y}{x^2}\right)\left(\frac{x}{1-x}\right) = \frac{y+3x^2}{x(1-x)}.$ At the point $(2, -2)$, $\frac{dy}{dx} = \frac{-2+3(2)^2}{2(1-2)} = -5$ 3. $x^2 + 3xy + y^2 = 7$ $\Rightarrow 2x + 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} = 0$ $\Rightarrow 2 + 3x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 3\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} +$ $2\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) = 0$ $\Rightarrow (3x+2y)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 6\frac{dy}{dx} + 2 = 0$ hence. Note that $\frac{d}{dx}[3x\frac{dy}{dx}]$ is differentiated as a product.
Summary (5minutes)	**Summary of learning points** - Some implicit functions can be rearranged to give explicit functions of one variable . - Product rule is very essential in differentiating	- Where possible rearrange an implicit function to give an explicit function of one variable e.g $xy - 2x = 3 \Rightarrow y = \frac{3+2x}{x}$ giving y explicitly in terms of x.

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	implicitly.	- $\frac{d}{dx}[\mathrm{f}(y)] = \mathrm{f}^{/}(x)\frac{dy}{dx}$ - To find $\frac{d}{dx}[(x^n)(y^m)]$ differentiate as product of x^n and y^m i.e $\frac{d}{dx}[(x^n)(y^m)] = (x^n)\frac{d}{dx}(y^m) + (y^m)\frac{d}{dx}(x^n)$ - Similarly $\frac{d}{dx}\left[\mathrm{f}^{/}(x)\frac{dy}{dx}\right] = \mathrm{f}^{/}(x)\frac{d^2y}{dx^2} + \mathrm{f}^{//}(x)\frac{dy}{dx}$.
Review of problem situation (10minutes)	Can you now help Afeminui out of her frustration? Given that $x + 3xy - y^3 = 5$, show that $$(x - y^2)\frac{d^2y}{dx^2} - 2y\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) = 0$$	**Solution of Afeminui's problem** $x + 3xy - y^3 = 5$ $\Rightarrow 1 + 3x\frac{dy}{dx} + 3y - 3y^2\frac{dy}{dx} = 0$ $\Rightarrow 0 + 3x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 3\frac{dy}{dx} - 3y^2\frac{d^2y}{dx^2} -$ $6y\frac{dy}{dx}\left(\frac{dy}{dx}\right) = 0$ $\Rightarrow 3(x - y^2)\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 6\left(\frac{dy}{dx}\right) = 0$ $\Rightarrow (x - y^2)\frac{d^2y}{dx^2} - 2y\left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) = 0$ hence.
Conclusion (5minutes)	**Assignment** Copy the assignment in your books and do them at home. Our next class is on differentiation of trigonometric functions	**Assignment** 1. Differentiate the following (a) implicitly with respect to x (b) by expressing y explicitly in terms of x and (c) by expressing x explicitly in terms of y. Show that your results for (a) , (b) and (c) are equivalent. (i) $xy = 5$ (ii) $y^2 = 8x$ 2. Find the gradient of the curve $2x^2 - 5y^2 - 3xy = 0$ at the point $(-1, 1)$. 3. Given that $x^2 + y^2 - 3x - 4y - 5 = 0$ show that $(y - 2)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$